On towers approximating homological localizations

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Abstract

Our object of study is the natural tower which, for any given map $f\colon A \to B$ and each space X, starts with the localization of X with respect to f and converges to X itself. These towers can be used to produce approximations to localization with respect to any generalized homology theory E_*, yielding e.g. an analogue of Quillen's plus-construction for each E_*. We discuss in detail the case of ordinary homology with coefficients in $\mathbf{Z}/p$ or $\mathbf{Z}[1/p]$. Our main tool is a comparison theorem for nullification functors (that is, localizations with respect to maps of the form $f\colon A \to \mathrm{pt}$), which allows us, among other things, to generalize Neisendorfer's observation that p-completion of simply-connected spaces coincides with nullification with respect to a Moore space $M(\mathbf{Z}[1/p], 1)$.

0 Introduction

If E_* is any generalized homology theory, then, as shown by Bousfield in [4], one can construct an idempotent functor $(\quad)_E$ on the pointed homotopy category of CW-complexes **Ho**, which renders invertible precisely the class of E-equivalences. Thus, a map $X \to Y$ induces an isomorphism $E_*(X) \cong E_*(Y)$ if and only if it induces a homotopy equivalence $X_E \simeq Y_E$. This functor is referred to as (unstable) E-localization, and it is fully understood only in a few special cases, often restricted to suitable subcategories of **Ho**.

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In the case when $E = HR$, ordinary homology with coefficients in a commutative ring R with unit, the HR-localization functor is closely related with the R-completion functor R_∞ defined by Bousfield and Kan in [9]. Specifically, there is a natural map $X_{HR} \to R_\infty X$ which is a homotopy equivalence in many cases, e.g. whenever X is nilpotent or, more generally, whenever X_{HR} is nilpotent. In fact, HR-localization is the best idempotent approximation to R-completion, in the sense that they render invertible precisely the same class of maps.

The present paper originated from the observation, made from various sources, that inverting one single easy map often produces the same effect —or almost the same effect— as inverting a whole family of homology equivalences. With the phrase "inverting one map" we allude to the recently developed theory of homotopical localization, which started from the work of Bousfield [6], [7] and Dror Farjoun [13]. One first instance is Neisendorfer's remark [21] that inverting a map of the form $M(\mathbf{Z}[1/p], 1) \to \mathrm{pt}$ (where "pt" denotes a one-point space and the letter M stands throughout for Moore spaces) has the same effect as $H\mathbf{Z}/p$-localization on simply-connected spaces. Similarly, by inverting the pth power map $p\colon S^1 \to S^1$, one obtains a functor which is closely related to $H\mathbf{Z}[1/p]$-localization, and in fact agrees with $H\mathbf{Z}[1/p]$-localization on nilpotent spaces [11]. Almost the same effect is obtained by killing the homotopy cofibre of p, that is, by inverting the map $M(\mathbf{Z}/p, 1) \to \mathrm{pt}$; see [2], [10]. Now, by inverting the suspension Σp one obtains exactly the same result as by inverting all $H\mathbf{Z}[1/p]$-equivalences which are suspensions, namely a functor which preserves the fundamental group and tensors with $\mathbf{Z}[1/p]$ the higher homotopy groups. If we continue this process of successively suspending p and killing the corresponding homotopy cofibres, we obtain a tower of localizations with increasingly weaker effect and converging to X. Of course, this makes sense for every map $f\colon A \to B$, and we call the resulting tower the *f-tower* associated to each space X. In the special case when f is the map $S^0 \to \mathrm{pt}$, the f-tower is precisely the Postnikov tower. Sufficient motivation for the study of such f-towers was offered in [7] and [15]. Note that, if we choose f to be a v_n-map for $n \geq 1$, then the outcome is a family of approximations to localization with respect to certain Morava K-theories. In particular, it would be interesting to decide how closely inverting a v_1-map approximates localization with respect to mod p complex K-theory; see [13].

It is important to retain the above suggested duality between $\mathbf{Z}/p$ and $\mathbf{Z}[1/p]$, which is very much in the spirit of some parts of the book by Bousfield and Kan [9]. Indeed, the most substantial part of our discussion involves reducing the study of certain localization functors to the study of their effect on the fundamental group of spaces. Thus we are led to comparing, for specific classes of groups, the effect of dividing out p-torsion (resp. all infinitely p-divisible elements) with the effect of dividing out the $\mathbf{Z}[1/p]$-perfect radical (resp. the $\mathbf{Z}/p$-perfect radical).

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1 Preliminaries

We assume that all spaces are pointed, and all maps and homotopies preserve basepoints. The space of based maps from X to Y is denoted by $\mathrm{map}_*(X, Y)$.

Let $f\colon A \to B$ be any map. We follow the terminology of [7], [13], by calling *f-local* those spaces X for which the induced map

$$f^*\colon \mathrm{map}_*(B, X) \to \mathrm{map}_*(A, X)$$

is a weak homotopy equivalence. Likewise, a map $h\colon X \to Y$ is an *f-equivalence* if

$$h^*\colon \mathrm{map}_*(Y, Z) \to \mathrm{map}_*(X, Z)$$

is a weak homotopy equivalence for every f-local space Z.

We recall from [6], [13] that, replacing if necessary the map f with a cofibration, one may construct a functor L_f on the pointed category of CW-complexes, together with a natural transformation $l\colon \mathrm{id} \to L_f$, such that, for every space X, the map $l_X\colon X \to L_f X$ is an f-equivalence and the space $L_f X$ is f-local. Moreover, L_f also defines a functor on the pointed homotopy category $\mathbf{Ho}$, and, if so viewed, then it is idempotent. As such, L_f is left adjoint to the inclusion of the full subcategory of f-local spaces in $\mathbf{Ho}$. It also follows that a map $h\colon X \to Y$ is an f-equivalence if and only if the induced map $L_f h\colon L_f X \to L_f Y$ is a homotopy equivalence, and this is the same as asserting that $f^*\colon [Y, Z] \to [X, Z]$ is bijective for all f-local spaces Z; cf. [1].

For simplicity, we use the term "idempotent functor" (in any category) to denote a functor L which is part of an idempotent monad (L, l), as in [1]. All natural transformations $\theta\colon L_2 \to L_1$ between idempotent functors are implicitly assumed to satisfy $\theta \cdot l_2 = l_1$. If L is an idempotent functor, we call *L-local* those objects X such that $X \cong LX$ and *L-equivalences* those morphisms φ such that $L\varphi$ is invertible. There is a natural transformation $\theta\colon L_2 \to L_1$ of idempotent functors if and only if the class of L_2-equivalences is contained in the class of L_1-equivalences, or, equivalently, the class of L_1-local objects is contained in the class of L_2-local objects; in this case, the natural transformation θ is unique. As a special case of this general situation, we infer the following.

Proposition 1.1 *Let f_1, f_2 be any two maps between spaces. Then there is a natural transformation $\theta\colon L_{f_2} \to L_{f_1}$ in* **Ho** *if and only if f_2 is an f_1-equivalence. Moreover, in this case, the natural transformation θ is unique.* $\square$

We emphasize another elementary fact which will be useful in the analysis of diagrams (4.1) and (4.2) below.

Proposition 1.2 *Let $L_3 \xrightarrow{\beta} L_2 \xrightarrow{\alpha} L_1$ be natural transformations of functors, where at least L_3 and L_2 are idempotent. Suppose that, for a certain object X, the composite arrow $L_3X \to L_1X$ is an isomorphism. Then both α_X and β_X are isomorphisms.*

Proof. From the assumption made, it follows that L_3X is a retract of L_2X and therefore L_3X is L_2-local. Hence the arrow $\beta_X\colon L_3X \to L_2X$ is an isomorphism, and so is also $\alpha_X\colon L_2X \to L_1X$. $\square$

If the target space B of the map $f : A \to B$ is contractible, then f-local spaces have been called *A-periodic* [7] or *A-null* [8]. The corresponding f-localization functor is referred to as *A-periodization* or *A-nullification*, and denoted by P_A. Thus, a space X is A-null if and only if $\mathrm{map}_*(A, X)$ is weakly contractible, or, equivalently, if $[\Sigma^k A, X]$ consists of a single element for all $k \geq 0$.

Proposition 1.3 [7], [14] *Let $F \to E \to X$ be a homotopy fibration and A any space. If X is A-null and connected, then $P_AF \to P_AE \to X$ is a homotopy fibration.* $\square$

Most of the above machinery can be paralleled in the category of groups. In fact, as explained in [6], homotopy localization with respect to a map (or a set of maps —there is no increase in generality in saying this, provided that coproducts exist in our category) makes sense in every cocomplete closed simplicial model category [22] satisfying certain mild conditions. Thus the category of groups (endowed with the discrete simplicial model structure) fits into this framework very well. Specifically, if $\varphi\colon G \to K$ is any group homomorphism, we say that a group L is *φ-local* if the induced map

$$\varphi^*\colon \mathrm{Hom}(K, L) \to \mathrm{Hom}(G, L)$$

is a bijection of sets. Then, by [6] or [12], one may construct, for every group π, a homomorphism $l_\pi : \pi \to L_\varphi \pi$ which is initial among all homomorphisms from π into φ-local groups. Thus (L_φ, l) may be viewed as an idempotent monad on the category of groups. We shall refer to it as *φ-localization*. In the special case when the target K of $\varphi\colon G \to K$ is the trivial group, we preferably use the term *G-reduction* instead of φ-localization, and call the corresponding φ-local groups *G-reduced*, motivated by the special case considered in [7, § 5]. The G-reduction map will be denoted, as proposed in [7], by $\pi \to \pi//G$. This is always an epimorphism [10, Theorem 3.2], and its kernel is called the *G-radical* of π. Thus, the G-radical of a group π is the (unique) maximal subgroup of π among those whose own G-reduction is trivial. The basic examples are the standard p-radical, where p is a prime (this case corresponds to $G = \mathbf{Z}/p$, and coincides with the p-torsion subgroup whenever such a subgroup exists), and the perfect radical (for which G can be chosen as the free product of a set of representatives of all isomorphism classes of countable perfect groups [3]).

The analogue of Proposition 1.1 is also true in the category of groups. Moreover, the following stronger result holds in the case of G-reductions:

Proposition 1.4 *Given two groups G_1, G_2, consider the homomorphisms $\varphi_i\colon G_i \to \{1\}$, $i = 1, 2$. Then the following assertions are equivalent:*

(a) *There is a natural transformation $\theta\colon L_{\varphi_2} \to L_{\varphi_1}$.*

(b) *φ_2 is a φ_1-equivalence.*

(c) *$G_2//G_1$ is the trivial group.*

(d) *For every group π, the G_2-radical of π is contained in the G_1-radical of π.*

(e) *The class of groups annihilated by L_{φ_2} is contained in the class of groups annihilated by L_{φ_1}.*

Moreover, if these equivalent assertions hold, then the natural transformation θ is unique, and θ_π is surjective for every group π. $\square$

2 On π_1-compatible nullifications

Let $f\colon A \to B$ be any map between connected spaces, and let $f_*\colon \pi_1 A \to \pi_1 B$ be the induced homomorphism of fundamental groups. Then, as shown in [10, § 3], for every connected space X the natural homomorphism $\pi_1 X \to \pi_1(L_f X)$ is an f_*-equivalence of groups and hence there is a natural transformation of functors

$$\pi_1 L_f \to L_{f_*}\pi_1. \tag{2.1}$$

We say that L_f is π_1-*compatible* if (2.1) is an isomorphism for all connected spaces X. That is, L_f is π_1-compatible if and only if the group $\pi_1(L_f X)$ is f_*-local for every connected space X.

If both A and B are (possibly infinite) wedges of circles, then, for every connected space X, the sets $[A, L_f X]$ and $[B, L_f X]$ are in natural bijective correspondence with the sets $\mathrm{Hom}(\pi_1 A, \pi_1(L_f X))$ and $\mathrm{Hom}(\pi_1 B, \pi_1(L_f X))$, respectively. It follows that the group $\pi_1(L_f X)$ is f_*-local and therefore the functor L_f is π_1-compatible. This observation, together with Theorem 3.5 in [10], yields

Theorem 2.1 *If $f\colon A \to B$ is any map between wedges of circles, then the functor L_f is π_1-compatible. If A is a CW-complex of dimension 1 or 2, then the A-nullification functor P_A is π_1-compatible.* $\square$

However, if $A = K(\mathbf{Z}/p, 1)$, then P_A is not π_1-compatible; see Example 3.4 in [10].

The properties of π_1-compatible nullification functors turn out to be particularly pleasant. For example, such functors are essentially determined by their effect on the fundamental group and their behaviour on simply-connected spaces. In the rest of this section, this claim is made more precise.

Theorem 2.2 *Let L be any idempotent functor on* **Ho**. *Suppose given a π_1-compatible nullification functor P_A together with a natural transformation $\theta\colon P_A \to L$ which is an equivalence on simply-connected spaces. Then θ is an equivalence on the class of spaces whose fundamental group is annihilated under $\pi_1 A$-reduction.*

Proof. Let X be any space such that $\pi_1 X//\pi_1 A$ is trivial. Then $P_A X$ is simply-connected, and therefore the lower horizontal arrow in the commutative diagram

$$\begin{array}{ccc} P_A X & \xrightarrow{\theta} & LX \\ \downarrow & & \downarrow \\ P_A(P_A X) & \xrightarrow{\theta} & L(P_A X) \end{array}$$

is a homotopy equivalence. By the idempotence of P_A, the left-hand vertical arrow is also a homotopy equivalence. Finally, the existence of θ ensures that every P_A-equivalence is an L-equivalence, so that the right-hand vertical arrow is a homotopy equivalence as well, from which our claim follows. $\square$

This provides us with a good method to decide whether two π_1-compatible nullification functors are isomorphic or not. Namely, we infer the following criterion, which will be used strongly in the special cases discussed in Section 4; cf. Theorem 4.5 below.

Theorem 2.3 *Assume given two spaces A_1, A_2 together with a natural transformation $\theta\colon P_{A_2} \to P_{A_1}$. Suppose further that both P_{A_2} and P_{A_1} are π_1-compatible, and that θ is an equivalence on simply-connected spaces. Then, for a space X, the map $\theta_X\colon P_{A_2}X \to P_{A_1}X$ is a homotopy equivalence if and only if the $\pi_1 A_1$-radical and the $\pi_1 A_2$-radical of $\pi_1 X$ coincide.*

Proof. Suppose that θ_X is a homotopy equivalence. Then the induced homomorphism $\pi_1(P_{A_2}X) \to \pi_1(P_{A_1}X)$ is an isomorphism. Since P_{A_1} and P_{A_2} are π_1-compatible, this means that $\pi_1 X//\pi_1 A_2 \cong \pi_1 X//\pi_1 A_1$, from which it follows that the $\pi_1 A_1$-radical and the $\pi_1 A_2$-radical of $\pi_1 X$ coincide.

Conversely, suppose that the radicals coincide. Let $\pi = \pi_1(P_{A_2}X) \cong \pi_1(P_{A_1}X)$ and let $\tilde{X}$ be the homotopy fibre of the natural map $X \to B\pi$. Since $\mathrm{Hom}(\pi_1 A_1, \pi)$ and $\mathrm{Hom}(\pi_1 A_2, \pi)$ are trivial, the space $B\pi$ is A_1-null and A_2-null. By Proposition 1.3, the rows of the commutative diagram

$$\begin{array}{ccccc} P_{A_2}\tilde{X} & \longrightarrow & P_{A_2}X & \longrightarrow & B\pi \\ \downarrow \theta_{\tilde{X}} & & \downarrow \theta_X & & \downarrow = \\ P_{A_1}\tilde{X} & \longrightarrow & P_{A_1}X & \longrightarrow & B\pi \end{array}$$

are homotopy fibrations. By Theorem 2.2, $\theta_{\tilde{X}}$ is a homotopy equivalence. Therefore, θ_X is also a homotopy equivalence. $\square$

Essentially the same argument proves the next result. This is in fact a broad generalization of Theorem 4.4 of [10], according to which, for every connected space X and every prime p, there is a homotopy fibration $(\mathbf{Z}[1/p])_\infty \tilde{X} \to P_{M(\mathbf{Z}/p,1)}X \to B\pi$, where π is the quotient of $\pi_1 X$ by its p-radical, and $\tilde{X}$ is the homotopy fibre of the map $X \to B\pi$. Another important instance of Theorem 2.4 is the case when L is ordinary homology localization and P_A is the associated plus-construction (see Section 3 below).

Theorem 2.4 *Let L be any idempotent functor on* **Ho**. *Suppose given a π_1-compatible nullification functor P_A together with a natural transformation $\theta\colon P_A \to L$ which is an equivalence on simply-connected spaces. For a given space X, write $\pi = \pi_1 X//\pi_1 A$, and let $\tilde{X}$ be the homotopy fibre of the natural map $X \to B\pi$. Then there is a commutative diagram*

$$\begin{array}{ccccc} \tilde{X} & \longrightarrow & X & \longrightarrow & B\pi \\ \downarrow & & \downarrow & & \downarrow = \\ L\tilde{X} & \longrightarrow & P_A X & \longrightarrow & B\pi \end{array}$$

in which the lower row is also a homotopy fibration. $\square$

3 Generalized Postnikov towers

Let $f : A \to B$ be any map (which we replace with a cofibration, if necessary). We consider the cofibre sequence

$$A \xrightarrow{f} B \to C \to \Sigma A \xrightarrow{\Sigma f} \Sigma B \to \Sigma C \to \Sigma^2 A \to \dots .$$

Thus we may view

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ \mathrm{pt} & \xrightarrow{g} & C \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{g'} & \mathrm{pt} \\ \downarrow & & \downarrow \\ \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \end{array}$$

as homotopy cofibre squares, showing that g is an f-equivalence and Σf is a g'-equivalence. In this situation, Proposition 1.1 yields natural transformations $L_{\Sigma f} \to P_C \to L_f$. Therefore, one obtains, for every space X, an inverse system

$$L_f X \leftarrow P_C X \leftarrow L_{\Sigma f} X \leftarrow P_{\Sigma C} X \leftarrow L_{\Sigma^2 f} X \leftarrow \dots, \tag{3.1}$$

which we call the *f-tower* associated to X. It is very often convenient to regard it, not as a single tower, but as two towers whose terms alternate, namely

$$L_f X \leftarrow L_{\Sigma f} X \leftarrow L_{\Sigma^2 f} X \leftarrow \dots \tag{3.2}$$

and

$$P_C X \leftarrow P_{\Sigma C} X \leftarrow P_{\Sigma^2 C} X \leftarrow \dots . \tag{3.3}$$

Of course, if the target B of f is contractible, then (3.3) is just a shift of (3.2). In the special case $f\colon S^0 \to \mathrm{pt}$, one obtains, up to homotopy, the Postnikov tower of X. Indeed, if we consider the map $\sigma_{n+1}\colon S^{n+1} \to \mathrm{pt}$, then the class of σ_{n+1}-equivalences is precisely the class of maps $X \to Y$ inducing isomorphisms $\pi_k X \cong \pi_k Y$ for $k \leq n$.

If $f\colon A \to B$ is any map between n-connected spaces, then f is a σ_{n+1}-equivalence. Therefore, according to our preliminary remarks in Section 1, every f-equivalence is a σ_{n+1}-equivalence. This gives an easy proof of the following statement (which is quite well-known, and could, in fact, be inferred from the construction of L_f described in [13] as well).

Proposition 3.1 *Let $f\colon A \to B$ be any map between n-connected spaces, $n \geq 1$. Then, for any connected space X, the f-localization map $l\colon X \to L_f X$ induces isomorphisms*

$$\pi_k X \cong \pi_k(L_f X) \qquad \text{for } k \leq n. \qquad \square$$

In addition, if A is n-connected and B is contractible, then we have an epimorphism

$$\pi_{n+1} X \twoheadrightarrow \pi_{n+1}(L_f X),$$

as deduced from the fact that the A-nullification of a space X is constructed by attaching copies of A and its suspensions $\Sigma^k A$ to X; cf. [7, Proposition 2.9].

As a consequence of Proposition 3.1, for every space X and every map $f: A \to B$, the f-tower of X converges to X, in the sense that the homotopy inverse limit of (3.1) is homotopy equivalent to X.

Theorem 3.2 *Let f_1, f_2 be two maps, and assume that f_2 is an f_1-equivalence. Then there is a unique commutative diagram of idempotent functors*

$$\begin{array}{ccccccccccc} L_{f_1} & \leftarrow & P_{C_1} & \leftarrow & L_{\Sigma f_1} & \leftarrow & P_{\Sigma C_1} & \leftarrow & L_{\Sigma^2 f_1} & \leftarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ L_{f_2} & \leftarrow & P_{C_2} & \leftarrow & L_{\Sigma f_2} & \leftarrow & P_{\Sigma C_2} & \leftarrow & L_{\Sigma^2 f_2} & \leftarrow & \dots . \end{array} \quad \square$$

Our main examples of f-towers involve homology localizations and certain "approximations" to these, as discussed in the next section. For a commutative ring R with unit and a space X, we denote by X_{HR} the localization of X with respect to ordinary homology with R coefficients; see [4]. More generally, let E_* be any homology theory satisfying the limit axiom. Let c be the smallest infinite cardinal which is at least equal to the cardinality of $E_*(\mathrm{pt})$. Let $h\colon A \to B$ be the wedge of a set of representatives of all homotopy classes of CW-inclusions which are E-equivalences and for which the cardinality of the set of cells of B is less than or equal to c. Then L_h is precisely E-localization; see [4, § 11] or [15]. Moreover, if C is the cofibre of h, then the following statements are equivalent:

- X is E-acyclic.
- X_E is contractible.

- $P_C X$ is contractible.

We shall use the notation X_E^+ for $P_C X$, and call it a *generalized plus-construction*. Indeed, if $E = H\mathbf{Z}$, then X_E^+ is homotopy equivalent to the Quillen plus-construction on X; cf. [3]. More generally, if $E = HR$ for a commutative ring R with unit, then X_{HR}^+ is homotopy equivalent to the *partial R-completion* of X defined in [9, VII.6]; see also [19], [20]. Recall that a group π is called *R-perfect* if $H_1(\pi; R) = 0$, and that every group G contains a unique maximal R-perfect normal subgroup $\mathcal{P}^R G$, which we call the *R-perfect radical* of G. The map $X \to X_{HR}^+$ is an HR-equivalence and induces the projection $\pi_1 X \to \pi_1 X / \mathcal{P}^R(\pi_1 X)$ on the fundamental group [9, VII.6.3].

Proposition 3.3 *For every commutative ring R with unit, the functors $(\quad)_{HR}$ and $(\quad)_{HR}^+$ are π_1-compatible.*

Proof. Let $h\colon A \to B$ be any map for which L_h is HR-localization. Then $h_*\colon \pi_1 A \to \pi_1 B$ induces an isomorphism on $H_1(\quad; R)$ and an epimorphism on $H_2(\quad; R)$, so that from [4, Lemma 7.3] it follows that $\pi_1(X_{HR})$ is h_*-local for every X. This means precisely that the functor $(\quad)_{HR}$ is π_1-compatible. Similarly, let C be the homotopy cofibre of h. Then $\pi_1 C$ is R-perfect. Hence, for every space X, the image of any homomorphism $\pi_1 C \to \pi_1 X$ is contained in $\mathcal{P}^R(\pi_1 X)$, which implies that $\pi_1(X_{HR}^+)$ is $\pi_1 C$-reduced, as required. $\square$

It would be very interesting to decide if $(\quad)_E$ and $(\quad)_E^+$ are π_1-compatible for other homology theories E_*, in particular for complex K-theory. Another challenging, closely related problem, is to find a purely group-theoretical description of the homomorphism $\pi_1 X \to \pi_1(X_E^+)$ in general.

In what follows, we shall use the notation

$$X_E \leftarrow X_E^+ \leftarrow X_{\Sigma E} \leftarrow X_{\Sigma E}^+ \leftarrow X_{\Sigma^2 E} \leftarrow \ldots \tag{3.4}$$

for the tower (3.1) in the case of E-localization.

4 Approximating ordinary homology towers

For any abelian group G, we may pick a space $M(G,1)$ as in [2, § 2]. Thus we choose a free abelian presentation

$$0 \to F_0 \to F_1 \to G \to 0$$

together with a map $w\colon W_0 \to W_1$ between suitable wedges of circles inducing the inclusion $F_0 \to F_1$ on homology, and define $M(G,1)$ to be the homotopy cofibre of w. Of course, if $n>1$, then $\Sigma^{n-1}M(G,1)$ is a Moore space $M(G,n)$ in the usual sense. If we apply the functor $[\ \ ,X]$ to the cofibre sequence associated with w, we obtain the following result; cf. [2] or [7, § 5].

Proposition 4.1 *Let G be any abelian group, and choose $w\colon W_0 \to W_1$ as above. Then, for a connected space X and $n>1$, we have:*

(a) *X is $\Sigma^{n-1}w$-local if and only if* $\mathrm{Ext}(G,\pi_m X) = \mathrm{Hom}(G,\pi_m X) = 0$ *for $m \geq n$.*

(b) *X is $M(G,n)$-null if and only if* $\mathrm{Ext}(G,\pi_m X) = \mathrm{Hom}(G,\pi_m X) = 0$ *for $m>n$ and $\pi_n X$ is G-reduced.*

Moreover, assertion (b) *is true for $n=1$ if $\pi_1 X$ is abelian, and assertion* (a) *is true for $n=1$ if X is simply-connected.* □

Let $M(\mathbf{Z}/p,1)$ be the homotopy cofibre of the standard map $f\colon S^1 \to S^1$ of degree p. On the other hand, by writing

$$\mathbf{Z}[1/p] = \langle x_1, x_2, \ldots, x_n, \ldots \mid x_i = x_{i+1}^p \quad \text{for all } i \rangle,$$

we have presented $\mathbf{Z}[1/p]$ as the cokernel of a specific homomorphism between free groups. Let $g\colon \vee_i S^1 \to \vee_i S^1$ induce this homomorphism, and denote by $M(\mathbf{Z}[1/p],1)$ the homotopy cofibre of g. Observe that the fundamental group of this space $M(\mathbf{Z}[1/p],1)$ is precisely $\mathbf{Z}[1/p]$, so that it is a "true" Moore space in the sense of [24]. From now on, the letters f and g will consistently denote the two maps which we have just considered.

Remark 4.2 When $G = \mathbf{Z}/p$, the condition that an abelian group A be $\mathbf{Z}/p$-reduced amounts to A being p-torsionfree, and the condition $\mathrm{Ext}(\mathbf{Z}/p,A) = \mathrm{Hom}(\mathbf{Z}/p,A) = 0$

says precisely that A is a p'-local group [17], where p' denotes the complement of p (thus, a p'-local abelian group is just a $\mathbf{Z}[1/p]$-module). In the case $G = \mathbf{Z}[1/p]$, for an abelian group A, the condition $\mathrm{Ext}(\mathbf{Z}[1/p], A) = \mathrm{Hom}(\mathbf{Z}[1/p], A) = 0$ says precisely that A is Ext-p-complete; see [9, VI.3.4] or [23].

Taking into account the fact that, in any category, two idempotent functors with the same image class are necessarily isomorphic, we have just shown the following.

Theorem 4.3 *Assume that X is connected and $n > 0$, or that X is simply-connected and $n \geq 0$. Then, if f and g denote the maps above defined, we have natural homotopy equivalences*

(a) $L_{\Sigma^n f} X \simeq X_{\Sigma^n H\mathbf{Z}[1/p]}$.

(b) $L_{\Sigma^n g} X \simeq X_{\Sigma^n H\mathbf{Z}/p}$.

(c) $P_{M(\mathbf{Z}/p, n+1)} X \simeq X^+_{\Sigma^n H\mathbf{Z}[1/p]}$.

(d) $P_{M(\mathbf{Z}[1/p], n+1)} X \simeq X^+_{\Sigma^n H\mathbf{Z}/p}$. $\square$

This result can be conveniently depicted as follows. Since

$$\tilde{H}_*(M(\mathbf{Z}/p, 1); \mathbf{Z}[1/p]) = 0,$$

Theorem 3.2 yields, for every space X, a homotopy commutative diagram of towers

$$\begin{array}{ccccccccccc}
\mathbf{Z}[1/p]_\infty X & \leftarrow & X_{H\mathbf{Z}[1/p]} & \leftarrow & X^+_{H\mathbf{Z}[1/p]} & \leftarrow & X_{\Sigma H\mathbf{Z}[1/p]} & \leftarrow & X^+_{\Sigma H\mathbf{Z}[1/p]} & \leftarrow & \cdots \\
 & & \uparrow \theta & & \uparrow \theta^+ & & \uparrow \simeq & & \uparrow \simeq & & \\
 & & L_f X & \leftarrow & P_{M(\mathbf{Z}/p,1)} X & \leftarrow & L_{\Sigma f} X & \leftarrow & P_{M(\mathbf{Z}/p,2)} X & \leftarrow & \cdots .
\end{array} \tag{4.1}$$

Similarly, the fact that $\tilde{H}_*(M(\mathbf{Z}[1/p], 1); \mathbf{Z}/p) = 0$ gives

$$\begin{array}{ccccccccccc}
(\mathbf{Z}/p)_\infty X & \leftarrow & X_{H\mathbf{Z}/p} & \leftarrow & X^+_{H\mathbf{Z}/p} & \leftarrow & X_{\Sigma H\mathbf{Z}/p} & \leftarrow & X^+_{\Sigma H\mathbf{Z}/p} & \leftarrow & \cdots \\
 & & \uparrow \zeta & & \uparrow \zeta^+ & & \uparrow \simeq & & \uparrow \simeq & & \\
 & & L_g X & \leftarrow & P_{M(\mathbf{Z}[1/p],1)} X & \leftarrow & L_{\Sigma g} X & \leftarrow & P_{M(\mathbf{Z}[1/p],2)} X & \leftarrow & \cdots .
\end{array} \tag{4.2}$$

According to Theorem 4.3, all vertical arrows in (4.1) and (4.2) are homotopy equivalences if X is simply-connected. In fact, in order to infer that θ^+ and ζ^+ are homotopy equivalences it suffices to impose that $\pi_1 X$ be abelian.

The extent to which $L_f X$ approximates $X_{H\mathbf{Z}[1/p]}$ when X is not simply-connected was carefully discussed in [11]. Among other things, it was shown that if X is nilpotent (or, more generally, if $L_f X$ is nilpotent) then θ is a homotopy equivalence [11, Proposition 8.1]. It seems very likely that similar results could be proved for the arrow ζ in (4.2), but this has not been done so far.

Theorem 4.4 *Let X be a connected space.*

(a) *If $\pi_1 X//(\mathbf{Z}/p) = \{1\}$, then the map $P_{M(\mathbf{Z}/p,1)}X \to \mathbf{Z}[1/p]_\infty X$ is a homotopy equivalence.*

(b) *If $\pi_1 X//\mathbf{Z}[1/p] = \{1\}$, then the map $P_{M(\mathbf{Z}[1/p],1)}X \to (\mathbf{Z}/p)_\infty X$ is a homotopy equivalence.*

Proof. We give the argument only for (b). By Theorem 2.1, the functor $P_{M(\mathbf{Z}[1/p],1)}$ is π_1-compatible. Hence, from Theorem 2.2 it follows that the arrows $P_{M(\mathbf{Z}[1/p],1)}X \to L_g X$ and $\zeta^+ \colon P_{M(\mathbf{Z}[1/p],1)}X \to X^+_{H\mathbf{Z}/p}$ are homotopy equivalences. Since, by assumption, the space $P_{M(\mathbf{Z}[1/p],1)}X$ is simply-connected, so is also $X^+_{H\mathbf{Z}/p}$, and hence $X^+_{H\mathbf{Z}/p} \to X_{H\mathbf{Z}/p}$ is also a homotopy equivalence. This ensures that X is $\mathbf{Z}/p$-good, so that the arrow $X_{H\mathbf{Z}/p} \to (\mathbf{Z}/p)_\infty X$ is also a homotopy equivalence. $\square$

This result improves substantially Lemma 1.2 in [21]. We devote the rest of this paper to giving other sufficient conditions on X under which θ^+ or ζ^+ are homotopy equivalences, and explicit counterexamples showing that sometimes they are not. The analysis turns out to be of a purely group-theoretical nature, due to the following facts. By Theorem 2.1 and Proposition 3.3, all idempotent functors in (4.1) and (4.2) are π_1-compatible. Hence, by Theorem 2.3, θ^+ and ζ^+ will be homotopy equivalences if and only if they induce isomorphisms on the fundamental group. In other words,

Theorem 4.5 *The map θ^+ is a homotopy equivalence if and only if the $\mathbf{Z}/p$-radical and the $\mathbf{Z}[1/p]$-perfect radical of $\pi_1 X$ coincide. The map ζ^+ is a homotopy equivalence if and only if the $\mathbf{Z}[1/p]$-radical and the $\mathbf{Z}/p$-perfect radical of $\pi_1 X$ coincide.* $\square$

The assumptions of Theorem 4.5 are satisfied in many cases. We record the following instances.

Theorem 4.6 *Suppose that π is free or nilpotent. Then:*

(a) *For every prime p, the $\mathbf{Z}/p$-radical and the $\mathbf{Z}[1/p]$-perfect radical of π coincide.*

(b) *For every prime p, the $\mathbf{Z}[1/p]$-radical and the $\mathbf{Z}/p$-perfect radical of π coincide.*

Proof. In case (a), if π is free, then both radicals are trivial; if π is nilpotent, then both radicals are equal to the p-torsion subgroup of π. In case (b), if π is free, then both radicals are again trivial. Now suppose that π is nilpotent. By Proposition 1.4, the existence of ζ^+ ensures that the $\mathbf{Z}[1/p]$-radical of π is contained in the $\mathbf{Z}/p$-perfect radical. Conversely, suppose that π contains a nontrivial subgroup G such that $H_1(G;\mathbf{Z}/p)=0$. We shall prove that $G//\mathbf{Z}[1/p]$ is trivial. Suppose the contrary, and write $Q=G//\mathbf{Z}[1/p]$ for shortness. Since Q is an epimorphic image of G, we have $H_1(Q;\mathbf{Z}/p)=0$, and this implies that $H_1(Q)$ is a p-divisible abelian group. If c is the nilpotency class of Q and Γ stands for the lower central series, then $\Gamma^{c-1}Q$ is nonzero and it is an epimorphic image of $\otimes^c H_1(Q)$; see [17, p. 44]. This implies that $\Gamma^{c-1}Q$ is p-divisible, in contradiction with the fact that Q is $\mathbf{Z}[1/p]$-reduced. $\square$

It follows, of course, that θ^+ and ζ^+ are homotopy equivalences whenever the fundamental group of X is free or nilpotent. Note that, if π is nilpotent, then the $\mathbf{Z}[1/p]$-radical of π is precisely the kernel of the Ext-p-completion map $\pi\to\mathrm{Ext}(\mathbf{Z}/p^\infty,\pi)$, as defined in [9, VI.3.7].

Theorem 4.7 *Suppose that π is finite. Then:*

(a) *For every prime p, the $\mathbf{Z}[1/p]$-radical and the $\mathbf{Z}/p$-perfect radical of π coincide.*

(b) *The $\mathbf{Z}/2$-radical and the $\mathbf{Z}[1/2]$-perfect radical of π coincide.*

Proof. In case (a), By Proposition 1.4, it suffices to prove that a finite group π is $\mathbf{Z}[1/p]$-reduced (i.e., $\mathrm{Hom}(\mathbf{Z}[1/p],\pi)=0$) if and only if it contains no nontrivial $\mathbf{Z}/p$-perfect subgroup. But both conditions are obviously equivalent to π being a p-group.

In case (b), the existence of θ^+ ensures that the $\mathbf{Z}/p$-radical of π is contained in the $\mathbf{Z}[1/p]$-perfect radical for all primes p (cf. Proposition 1.4). To prove the converse for $p = 2$, let Q be the quotient of π by its $\mathbf{Z}/2$-radical. Then Q has odd order. Suppose that Q contains a subgroup H which is $\mathbf{Z}[1/2]$-perfect. Then, since H has odd order, it is in fact $\mathbf{Z}$-perfect. On the other hand, by the Feit–Thompson theorem [16], H is solvable. This forces H to be trivial. Therefore, the $\mathbf{Z}/2$-radical of π contains all $\mathbf{Z}[1/2]$-perfect subgroups of π. $\square$

Example 4.8 Let π be any torsionfree perfect group. Then, for any prime p, the $\mathbf{Z}/p$-radical of π is trivial, while the $\mathbf{Z}[1/p]$-perfect radical is the whole of π. If $p \neq 2$, then we may even find examples of finite groups for which the $\mathbf{Z}/p$-radical and the $\mathbf{Z}[1/p]$-perfect radical are distinct, hence showing that part (b) of Theorem 4.7 fails for primes other than 2 in general. Namely, it suffices to pick a finite simple group whose order is not divisible by p (such groups are indeed available for all primes $p \neq 2$).

Example 4.9 For any fixed prime p, let π be an infinite perfect group of exponent p^r for some $r \geq 1$. Since π cannot contain any infinitely p-divisible element, it is $\mathbf{Z}[1/p]$-reduced, and hence the $\mathbf{Z}[1/p]$-radical of π is trivial. However, the $\mathbf{Z}/p$-perfect radical of π is the whole of π.

One might ask if such groups actually exist! We next indicate a source, which was suggested to us by Ian Leary. Let

$$B(m,n) = F_m/(F_m)^n$$

be a free Burnside group of exponent n, where F_m denotes a free group of rank m. For any prime p and any $m \geq 2$, we may choose $n = p^r$ big enough so that $B(m,n)$ is infinite [18]. After this choice, let π be the minimal normal subgroup of finite index in $B(m,n)$; see [25], [26]. Then π does not contain any proper normal subgroup of finite index, and hence the commutator subgroup of π has to be π itself. This shows that π is a perfect group of exponent a power of p, as desired.

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