Chain Complexes

Let R be a ring with 1. A *chain complex* of left R-modules is a sequence A_* of left R-modules and R-module homomorphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \longrightarrow \cdots$$

for $n \in \mathbb{Z}$, such that $d_n \circ d_{n+1} = 0$ for all n. If $A_n = 0$ for $n < 0$, then the chain complex A_* is called *positive*. The arrows d_n are called *differentials* or *boundaries*.

Morphisms of Chain Complexes

If A_* and B_* are chain complexes of left R-modules, a *morphism* $f_*\colon A_* \to B_*$ is a collection of R-module homomorphisms $f_n\colon A_n \to B_n$ for $n \in \mathbb{Z}$ such that $f_n \circ d^A_{n+1} = d^B_{n+1} \circ f_{n+1}$ for all n, where d^A_n denotes the nth differential of A_* and d^B_n denotes the nth differential of B_*.

Homology

If A_* is a chain complex of left R-modules, then the condition $d_n \circ d_{n+1} = 0$ implies that $\operatorname{Im} d_{n+1} \subseteq \operatorname{Ker} d_n$. The *homology* of A_* is the collection of R-modules defined as

$$\boxed{H_n(A_*) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}}$$

for all n. If the chain complex A_* is positive and $H_n(A_*) = 0$ for $n \neq 0$ then A_* is called *acyclic*. An acyclic complex A_* is also called a *resolution* of M where $M = H_0(A_*)$, and the epimorphism $A_0 \to M$ is called the *augmentation* of A_*.

Exercises

1. Let A_* denote the chain complex of $\mathbb{Z}$-modules in which $A_n = \mathbb{Z}/m$ for all n, where m is a fixed integer (possibly zero), and $d_n = 0$ if n is even while d_n is multiplication by k if n is odd, where k is another given integer:

$$\cdots \xrightarrow{0} \mathbb{Z}/m \xrightarrow{k} \mathbb{Z}/m \xrightarrow{0} \mathbb{Z}/m \xrightarrow{k} \mathbb{Z}/m \xrightarrow{0} \cdots$$

   Compute the homology groups $H_n(A_*)$ for all n.

2. Let G be a cylic group of order 2 and let $\mathbb{Z}G$ denote the group ring of G. Let A_* denote the positive chain complex of $\mathbb{Z}G$-modules in which $A_n = \mathbb{Z}G$ for all $n \geq 0$, and d_n is multiplication by $1+x$ if n is even while d_n is multiplication by $1-x$ if n is odd, where x denotes a generator of G.

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{1-x} \mathbb{Z}G \xrightarrow{1+x} \mathbb{Z}G \xrightarrow{1-x} \mathbb{Z}G \xrightarrow{1+x} \cdots \xrightarrow{1-x} \mathbb{Z}G$$

   Prove that A_* is acyclic and $H_0(A_*) \cong \mathbb{Z}$.

3. Compute the homology of the chain complex of $\mathbb{Z}$-modules

$$\mathbb{Z}^2 \xrightarrow{d_3} \mathbb{Z}^5 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z}$$

where

$$d_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & -2 \end{pmatrix}; \quad d_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 \\ 2 & 0 & 2 & 0 & 2 \end{pmatrix}; \quad d_1 = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}.$$

4. Compute the homology of the chain complex of $\mathbb{Z}$-modules

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}$$

where

$$d_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}; \quad d_1 = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

5. Prove that the following sequence is a chain complex of $\mathbb{Q}$-modules and find its homology:

$$\mathbb{Q}^2 \xrightarrow{d_5} \mathbb{Q}^2 \xrightarrow{d_4} \mathbb{Q}^2 \xrightarrow{d_3} \mathbb{Q}^2 \xrightarrow{d_2} \mathbb{Q}^2 \xrightarrow{d_1} \mathbb{Q}^2$$

where

$$d_5 = \begin{pmatrix} 5 & 1 \\ 0 & 0 \end{pmatrix}; \quad d_4 = \begin{pmatrix} 0 & 3 \\ 0 & -2 \end{pmatrix}; \quad d_3 = \begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix};$$

$$d_2 = \begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix}; \quad d_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

6. Prove that the following sequence is a chain complex of $\mathbb{Z}/2$-modules and find its homology:

$$(\mathbb{Z}/2)^2 \xrightarrow{d_3} (\mathbb{Z}/2)^3 \xrightarrow{d_2} (\mathbb{Z}/2)^3 \xrightarrow{d_1} (\mathbb{Z}/2)^2$$

where

$$d_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad d_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}; \quad d_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$