Poincaré Duality

Compact Support Cohomology

The *cohomology groups with compact support* of a manifold M are defined as

$$H_c^p(M) = \operatorname{colim}_K \, H^p(M, M \setminus K),$$

where the colimit is indexed by the directed system of all compact subspaces of M. Therefore, a class in $H_c^p(M)$ is represented by a relative cocycle in $S^p(M, M \setminus K)$ for some compact subspace K.

If M is compact, then $H_c^p(M) \cong H^p(M)$, since M is a maximal compact subspace of itself. If M is not compact, then $H_c^p(M)$ is the pth singular cohomology group of the Alexandroff compactification of M if $p \geq 1$. In particular,

$$H_c^p(\mathbb{R}^n) \cong H^p(S^n)$$

for $p \geq 1$, while $H_c^0(\mathbb{R}^n) = 0$.

Poincaré Duality Theorem

Suppose given an orientation on an n-dimensional topological manifold M. If K is a compact subspace of M, then the induced orientation on K is represented by a class

$$\zeta_K \in H_n(M, M \setminus K).$$

Then the cap product yields a homomorphism

$$(-) \frown \zeta_K \colon H^p(M, M \setminus K) \longrightarrow H_{n-p}(M)$$

for each p, which, by naturality, is compatible with the homomorphisms

$$H^p(M, M \setminus K) \longrightarrow H^p(M, M \setminus L)$$

whenever $K \subseteq L$. Passing to the colimit, we obtain a homomorphism

$$D_M \colon H_c^p(M) \longrightarrow H_{n-p}(M).$$

The *Poincaré Duality Theorem* asserts that, if M is orientable, then D_M is an isomorphism for all p. Moreover, if the assumption that M is orientable is omitted, then

$$D_M \colon H_c^p(M; \mathbb{Z}/2) \longrightarrow H_{n-p}(M; \mathbb{Z}/2)$$

is still an isomorphism for all p.

Betti Numbers

If M is compact, connected and orientable and we pick a generator ζ of $H_n(M)$, then the Poincaré Duality Theorem tells us that

$$H^p(M) \cong H_{n-p}(M) \tag{1}$$

for all p, where the isomorphism is given by $\alpha \mapsto \alpha \frown \zeta$.

The integers $\beta_i = \dim_{\mathbb{Q}} H_i(M;\mathbb{Q})$ are called *Betti numbers* of M. If M is compact, connected and orientable, then (1) implies that $\beta_i = \beta_{n-i}$ for all i, where n is the dimension of M. This is due to the fact that

$$H_i(M;\mathbb{Q}) \cong H^i(M;\mathbb{Q}) \cong H_{n-i}(M;\mathbb{Q}),$$

where the first isomorphism comes from Kronecker duality (using the fact that, if M is compact, then its homology groups are finitely generated) and the second isomorphism is given by Poincaré duality.

Signature

One of the most relevant outcomes of Poincaré duality is the signature of a manifold. Let M be compact, connected and orientable, and suppose that $\dim M = 4k$. Assume chosen an orientation class $\zeta \in H_{4k}(M)$. Then we have well-defined isomorphisms

$$H^{4k}(M;\mathbb{Q}) \cong H_{4k}(M;\mathbb{Q}) \cong \mathbb{Q},$$

given by Kronecker duality and by the choice of ζ. Then the cup product on the middle dimension

$$H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \longrightarrow H^{4k}(M;\mathbb{Q})$$

is a symmetric bilinear form, since the cup product is commutative in even degrees. This bilinear form is called *intersection form*, since it corresponds, via Poincaré duality, with transverse intersection of submanifolds.

The intersection form is nondegenerate, since if there is a class $\rho \in H^{2k}(M;\mathbb{Q})$ such that $\rho \smile \alpha = 0$ for all α, then

$$0 = \langle \rho \smile \alpha, \zeta \rangle = \langle \alpha, \rho \frown \zeta \rangle$$

for all α, and it follows that $\rho \frown \zeta = 0$. Since the Duality Theorem tells us that $(-) \frown \zeta$ is an isomorphism, we conclude that $\rho = 0$.

The signature of this nondegenerate symmetric bilinear form is called the *signature* of M. If the dimension of M is not a multiple of 4, then its signature is defined as zero.

Exercises

52. Prove that, for a manifold M and an open subspace $U \subseteq M$, the excision isomorphisms

$$H^p(U, U \setminus K) \cong H^p(M, M \setminus K)$$

for $K \subseteq U$ compact induce a homomorphism $H^p_c(U) \longrightarrow H^p_c(M)$ for each p. Prove that this homomorphism is not an isomorphism in general.

53. Prove the *Onion Lemma*: Let M be a topological manifold of dimension n and let $\mathcal{S}$ be a collection of open sets in M. Suppose that the following conditions hold:

    (a) If an open set U is homeomorphic to a convex open subset of $\mathbb{R}^n$, then $U \in \mathcal{S}$.

    (b) If two open sets U, V are in $\mathcal{S}$, and $U \cap V \in \mathcal{S}$, then $U \cup V \in \mathcal{S}$.

    (c) If $\{U_i\}_{i \in I}$ is a family of pairwise disjoint open sets of M such that $U_i \in \mathcal{S}$ for all $i \in I$, then $\cup_{i \in I} U_i \in \mathcal{S}$.

    Then it follows that $M \in \mathcal{S}$.

54. Prove the Poincaré Duality Theorem using the Onion Lemma.

55. Prove that, if M is a connected, compact, orientable manifold of dimension n, then $H_{n-1}(M)$ is a free abelian group.

56. Prove that, if M is a simply-connected compact manifold of dimension 3, then $H_i(M) \cong H_i(S^3)$ for all i.

57. Prove that, if M is a connected, compact, nonorientable manifold of dimension 3, then the fundamental group $\pi_1(M)$ is infinite.

58. The *Euler characteristic* of a compact manifold M is the alternating sum of its Betti numbers, $\chi(M) = \sum_{i=0}^{\infty} (-1)^i \beta_i$. Prove that if M is a connected compact manifold of odd dimension, then $\chi(M) = 0$.

59. (a) Prove that $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ for all n as graded rings, with $\deg x = 2$. Deduce from this fact that $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$.

    (b) Prove that $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/(x^{n+1})$ for all n as graded rings, with $\deg x = 1$. Deduce from this fact that $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]$.

60. (a) Find the signature of $S^1 \times \cdots \times S^1$ for an arbitrary (finite) number of factors.

    (b) Find the signature of $\mathbb{C}P^n$ for all n.