Differential Forms on Smooth Manifolds

Differential Forms

Let M be a topological manifold of dimension $n \geq 1$. For $k \geq 0$, a *smooth k-form* or *differential k-form* on M is a smooth section

$$\omega\colon M \longrightarrow T^*M \wedge \overset{(k)}{\cdots} \wedge T^*M$$

of the kth exterior power of the cotangent bundle $T^*M \to M$. Thus, at each point $p \in M$, we can view ω_p as an $\mathbb{R}$-multilinear map

$$\omega_p\colon T_pM \times \overset{(k)}{\cdots} \times T_pM \longrightarrow \mathbb{R}$$

which is *alternating*, that is $\omega_p(\dots, v_i, \dots, v_j \dots) = -\omega_p(\dots, v_j, \dots, v_i \dots)$ for all $i \neq j$.

If ω is a smooth k-form on M and $X_1, \dots, X_k$ are smooth vector fields on M, then $\omega(X_1 \dots, X_k)$ is a smooth function on M given by

$$(\omega(X_1 \dots, X_k))(p) = \omega_p((X_1)_p, \dots, (X_k)_p)$$

for every $p \in M$.

We denote by $\Omega^k(M)$ the $\mathbb{R}$-vector space of smooth k-forms on M. If (U, φ) is a chart in M with coordinates $\varphi = (x_1, \dots, x_n)$, then each form $\omega \in \Omega^k(M)$ can be written uniquely on U as

$$\sum_{i_1 < \cdots < i_k} \omega_{i_1 \dots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

with $i_r \in \{1, \dots, n\}$ for all r, and each $\omega_{i_1 \dots i_k}$ is a smooth function $U \to \mathbb{R}$.

Wedge Product

The graded $\mathbb{R}$-vector space $\Omega^*(M) = \oplus_{k=0}^{\infty} \Omega^k(M)$ can be given a graded ring structure as follows. Given $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, the *wedge product* $\omega \wedge \eta$ is the smooth $(k+l)$-form on M given, in coordinate notation, by

$$\omega \wedge \eta = \sum \omega_{i_1 \dots i_k}\, \eta_{j_1 \dots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}$$

followed by a suitable reordering of the terms, where $\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \dots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and $\eta = \sum_{j_1 < \cdots < j_l} \eta_{j_1 \dots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$.

The graded ring $\Omega^*(M)$ is anticommutative, that is,

$$\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega \quad \text{if } \omega \in \Omega^k(M) \text{ and } \eta \in \Omega^l(M).$$

Pull-back along a Smooth Map

If $f\colon M \to N$ is a smooth map between smooth manifolds and ω is a smooth k-form on N, then $f^*(\omega)$ is the smooth k-form on M given by

$$(f^*(\omega))_p(v_1,\dots,v_k) = \omega_{f(p)}((df)_p(v_1),\dots,(df)_p(v_k))$$

for every $p \in M$ and all $v_1,\dots,v_k \in T_pM$.

In coordinate notation, if $\omega = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$, then

$$f^*(\omega) = \sum_{i_1<\dots<i_k} (\omega_{i_1\dots i_k} \circ f)\, df_{i_1} \wedge \dots \wedge df_{i_k}$$

where f_j denotes the jth component of f in the given chart of N.

It then follows that $f^*\colon \Omega^*(N) \to \Omega^*(M)$ is a ring homomorphism, that is,

$$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta) \quad \text{for all } \omega,\ \eta.$$

Exterior Derivative

The *exterior derivative* is an $\mathbb{R}$-linear map $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ given in coordinate notation by

$$d\omega = \sum_{i_1<\dots<i_k} (d\omega_{i_1\dots i_k}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

if $\omega = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$, where $df = \sum_{j=1}^n (\partial f/\partial x_j)\, dx_j$.

This operator satisfies $d \circ d = 0$ and commutes with pull-back: $d(f^*(\omega)) = f^*(d\omega)$ for every $\omega \in \Omega^*(N)$ and every smooth map $f\colon M \to N$.

Therefore, for every smooth manifold M, we may view $\Omega^*(M)$ as a cochain complex, called the *de Rham complex* of M, and for each smooth map $f\colon M \to N$ the induced ring homomorphism $f^*\colon \Omega^*(N) \to \Omega^*(M)$ is also a homomorphism of cochain complexes.

De Rham Cohomology

The *de Rham cohomology* of a smooth manifold M is the cohomology of the cochain complex $\Omega^*(M)$. It is denoted by $H^*_{\mathrm{dR}}(M)$. Thus, a k-cocycle is a smooth k-form ω on M such that $d\omega = 0$. Such a form is called *closed*. A k-coboundary is a smooth k-form ω such that $\omega = d\eta$ for some smooth $(k-1)$-form η. If so, then ω is called *exact*, and η is a *primitive* of ω. Thus, the statement that all closed smooth k-forms are exact is equivalent to the statement that $H^k_{\mathrm{dR}}(M) = 0$.

Each smooth map $f\colon M \to N$ between smooth manifolds induces a well-defined ring homomorphism $f^*\colon H^*_{\mathrm{dR}}(N) \to H^*_{\mathrm{dR}}(M)$.

The *de Rham Theorem* states that

$$\boxed{H^*_{\mathrm{dR}}(M) \cong H^*(M;\mathbb{R})}$$

as graded rings for every smooth manifold M, where $H^*_{\mathrm{dR}}(M)$ is equipped with the wedge product and $H^*(M;\mathbb{R})$ is equipped with the cup product. The isomorphism is given as follows: if ω is a closed smooth k-form on M, then we assign to it the singular k-cochain sending each smooth k-simplex $\sigma\colon \Delta^k \to M$ to the value of the integral $\int_\sigma \omega$. This map is well defined by Stokes' Theorem.

Exercises

68. Prove the following statements:

    (i) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then $\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega$.

    (ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

    (iii) If $f\colon M \to N$ is a smooth map, then $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$ for all ω, η.

    (iv) If $f\colon M \to N$ is a smooth map, then $d(f^*(\omega)) = f^*(d\omega)$ for every ω.

69. Express the 2-form $x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$ in spherical coordinates (ρ, φ, θ), defined by
$$(x, y, z) = (\rho \cos\varphi \cos\theta,\ \rho \cos\varphi \sin\theta,\ \rho \sin\varphi).$$

70. For a smooth 2-form $\alpha = \sum_{i<j} \alpha_{ij}\, dx_i \wedge dx_j$, prove that $d\alpha = 0$ if and only if
$$\frac{\partial \alpha_{ij}}{\partial x_k} - \frac{\partial \alpha_{ik}}{\partial x_j} + \frac{\partial \alpha_{jk}}{\partial x_i} = 0 \quad \text{for all } i < j < k.$$

71. (a) Find a 1-form η on $\mathbb{R}^3$ such that $d\eta = (1 - x^2)\, dx \wedge dy + 3x^2\, dx \wedge dz - dy \wedge dz$.

    (b) Find a 2-form ν on $\mathbb{R}^3$ such that $d\nu = (xy - 2y^2 z)\, dx \wedge dy \wedge dz$.

72. Prove that the smooth 1-forms on $M = \mathbb{R}^2 \setminus \{(0,0)\}$ given by
$$\omega = \frac{x}{x^2 + y^2}\, dx + \frac{y}{x^2 + y^2}\, dy, \qquad \eta = \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$
are closed. Which of these are exact?

73. Prove that the smooth 2-form $x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$ is closed but not exact on the unit sphere $S^2 \subseteq \mathbb{R}^3$.

74. A smooth manifold is *orientable* if it admits a smooth atlas $\{(U_i, \varphi_i)\}_{i \in I}$ in which the differential of $\varphi_j \circ (\varphi_i)^{-1}$ has positive determinant at each point for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$. Prove that a smooth manifold is orientable if and only if the underlying topological manifold is orientable.

75. Prove that a smooth n-dimensional manifold M is orientable if and only if there is a smooth n-form ν such that $\nu_p \neq 0$ for all $p \in M$.

76. Prove that, if M is a topological manifold with boundary, then every smooth structure on M induces a smooth structure on the boundary ∂M, and if M is orientable then ∂M is also orientable.

77. Let M be a connected smooth manifold and let U, V be connected open subsets such that $M = U \cup V$. For a 1-form ω on M, prove that if the restriction of ω to U is exact and the restriction of ω to V is also exact and $U \cap V$ is connected, then ω is exact. Find a counterexample if the assumption that $U \cap V$ is connected is omitted.

78. Prove that, if M and N are smooth manifolds, then
$$H^*_{\mathrm{dR}}(M \times N) \cong H^*_{\mathrm{dR}}(M) \otimes H^*_{\mathrm{dR}}(N).$$

79. Using de Rham's Theorem, find the de Rham cohomology of each compact connected smooth surface.

80. Find generators of the de Rham cohomology of $\mathbb{C}P^n$ following the next steps:

    (a) Let (z_0, z_1, z_2) be complex homogeneous coordinates on $\mathbb{C}P^2$. View $\mathbb{C}P^1$ as the set of points of the form $(0, z_1, z_2)$ and parametrize its affine complement $\mathbb{A}$ as $(1, u, v)$ with $u, v \in \mathbb{C}$.

    (b) Use the coordinate change $u = re^{2\pi i\alpha}$, $v = se^{2\pi i\beta}$ on $\mathbb{A}$, with $r \geq 0$, $s \geq 0$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$.

    (c) Prove that r^2 is a smooth map on $\mathbb{A}$ (however, r is not).

    (d) Prove that the 1-form $r\,dr$ is smooth on $\mathbb{A}$.

    (e) Prove that the 1-form $r^2\,d\alpha$ and the 2-form $r\,dr \wedge d\alpha$ are smooth.

    (f) Prove that, if
    $$\eta = \frac{r^2\,d\alpha + s^2\,d\beta}{1 + r^2 + s^2},$$
    then $d\eta$ can be extended over a closed smooth 2-form ω on $\mathbb{C}P^2$. (*Hint:* Analyze how η changes when we move from one chart to another on $\mathbb{C}P^2$.)

    (g) Prove that $\displaystyle\int_{\mathbb{C}P^2} \omega \wedge \omega = 1$ and infer from this fact that ω is not exact.

    (h) Prove that ω is a generator of $H^2_{\mathrm{dR}}(\mathbb{C}P^2)$.

    (i) Prove that $\omega \wedge \cdots \wedge \omega$ (k times) is a generator of $H^{2k}_{\mathrm{dR}}(\mathbb{C}P^n)$ for every $k \leq n$.

81. The *Hopf invariant* of a smooth map $f\colon S^3 \to S^2$ is defined as
$$H(f) = \int_{S^3} \alpha \wedge f^*(\omega),$$
where ω is any smooth 2-form on S^2 with $\int_{S^2} \omega = 1$ and α is a primitive of $f^*(\omega)$, that is, $d\alpha = f^*(\omega)$.

    (a) Prove that $H(f)$ does not depend on the choices of ω and α.

    (b) Compute the Hopf invariant of the *Hopf fibration* $f\colon S^3 \to S^2$, which is given by
    $$f(x, y, z, t) = (x^2 + y^2 - z^2 - t^2,\ 2(xt - yz),\ 2(yt - xz)).$$

82. Prove that, if M is a smooth compact orientable n-dimensional manifold, then, for each $0 \leq k \leq n$, the map from $\Omega^k(M)$ to the dual of $\Omega^{n-k}(M)$ sending each $\omega \in \Omega^k(M)$ to the homomorphism $\eta \mapsto \int_M \omega \wedge \eta$ is an isomorphism, and hence
$$\dim H^k_{\mathrm{dR}}(M) \cong \dim H^{n-k}_{\mathrm{dR}}(M) \quad (\textit{Poincaré duality}).$$