# First Homology Group

## Fundamental Group

The *fundamental group* of a topological space $X$ with base point $x_0$ (also called *Poincaré group* or *first homotopy group*), denoted $\pi_1(X, x_0)$, is the group of homotopy classes of loops in $X$ starting and ending at $x_0$, operating under concatenation, where homotopies are assumed to be constant over $x_0$.

## Abelianization

For a group $G$, the *commutator* of two elements $x, y \in G$ is the element $xyx^{-1}y^{-1}$. The *commutator subgroup* of $G$ is the subgroup generated by all commutators of all pairs of elements of $G$. It is denoted by $[G, G]$. Note that $[G, G] = \{1\}$ if and only if $G$ is commutative.

Since $z(xyx^{-1}y^{-1})z^{-1} = (zxz^{-1})(zyz^{-1})(zxz^{-1})^{-1}(zyz^{-1})^{-1}$ for all $x$, $y$, $z$ in $G$, the commutator subgroup is *normal* in $G$, that is, closed under conjugation. Therefore, the quotient

$$G_{\mathrm{ab}} = G/[G, G]$$

acquires a group structure. It is called the *abelianization* of $G$ and it is the largest quotient of $G$ which is a commutative group.

## Hurewicz–Poincaré Theorem

If $X$ is any path-connected topological space and $x_0$ is any point of $X$, then

$$\boxed{H_1(X) \cong \pi_1(X, x_0)_{\mathrm{ab}}.}$$

## Exercises

12. Prove that the canonical projection $\alpha\colon G \to G_{\mathrm{ab}}$ is characterized up to isomorphism by the following *universal property*: $\alpha$ is a group homomorphism onto an abelian group, and for every homomorphism $\varphi\colon G \to A$ where $A$ is abelian there is a unique homomorphism $\psi\colon G_{\mathrm{ab}} \to A$ such that $\psi \circ \alpha = \varphi$.

13. Choose a homeomorphism between $\Delta^1$ and the interval $[0, 1]$, and prove that, if $\omega$ and $\sigma$ are homotopic loops at a point $x_0$ of a space $X$, then $\omega$ and $\sigma$ are homologous if viewed as 1-cycles. Show that this fact yields a well-defined group homomorphism $\varphi\colon \pi_1(X, x_0) \to H_1(X)$, which is surjective.

14. Compute the first homology group of all compact connected surfaces.

15. Let $X$ be a finite connected (unoriented) graph with $p$ vertices and $q$ edges. Prove that $H_1(X)$ is a free abelian group and find its rank.