Diagram Chasing

Five Lemma

Suppose given a commutative diagram of R-module homomorphisms

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5.
\end{tikzcd}$$

Suppose that both rows are exact and h_2 and h_4 are isomorphisms, h_1 is surjective and h_5 is injective. Then h_3 is an isomorphism.

Snake Lemma

Suppose given a commutative diagram of R-module homomorphisms

$$\begin{tikzcd}
 & A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & B_1 \arrow[r, "g_1"] \arrow[d, "h_2"] & C_1 \arrow[r] \arrow[d, "h_3"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "f_2"] & B_2 \arrow[r, "g_2"] & C_2. &
\end{tikzcd}$$

If both rows are exact, then there is a homomorphism $\Delta\colon \operatorname{Ker} h_3 \to \operatorname{Coker} h_1$ (called *connecting homomorphism*) such that the following sequence is exact:

$$\operatorname{Ker} h_1 \xrightarrow{f_1} \operatorname{Ker} h_2 \xrightarrow{g_1} \operatorname{Ker} h_3 \xrightarrow{\Delta} \operatorname{Coker} h_1 \xrightarrow{\bar{f}_2} \operatorname{Coker} h_2 \xrightarrow{\bar{g}_2} \operatorname{Coker} h_3.$$

Nine Lemma

Suppose given a commutative diagram of R-module homomorphisms

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_3 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Suppose that all columns are exact. If the two bottom rows are exact, then the top row is exact as well, and if the two top rows are exact, then the bottom row is exact.

Exercises

16. Prove the five lemma and give examples showing that h_3 need not be an isomorphism if either h_1 fails to be surjective or h_5 fails to be injective.

17. Prove the snake lemma and deduce from it that every short exact sequence of chain complexes $0 \to A_* \to B_* \to C_* \to 0$ gives rise to a long exact sequence of homology groups

$$\cdots \to H_n(A_*) \to H_n(B_*) \to H_n(C_*) \to H_{n-1}(A_*) \to H_{n-1}(B_*) \to \cdots .$$

18. Give an example showing that, for a pair (X, A) of topological spaces, the exact sequence of singular chain complexes

$$0 \longrightarrow S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X, A) \longrightarrow 0$$

does not split in general, although the exact sequence of abelian groups

$$0 \longrightarrow S_n(A) \longrightarrow S_n(X) \longrightarrow S_n(X, A) \longrightarrow 0$$

splits for every n.

19. Infer the nine lemma from the snake lemma and show by means of an example that, if the columns are exact and the top and bottom rows are exact, then the middle row need not be exact. However, if the top and bottom rows are exact and the composite $A_2 \to B_2 \to C_2$ is zero, then the middle row is exact.