Universal Coefficient Formulas

For a topological space X and an integer $n \geq 0$, let $S_n(X)$ denote the free abelian group on the set of continuous maps $\Delta^n \to X$, where Δ^n is the standard n-simplex, and let $d_i\colon \Delta^{n-1} \to \Delta^n$ be the ith face inclusion for $0 \leq i \leq n$.

Homology and Cohomology with Coefficients

Let G be any abelian group, and denote

$$S_n(X;G) = G \otimes S_n(X); \qquad S^n(X;G) = \mathrm{Hom}(S_n(X), G).$$

The elements of $S_n(X;G)$ are called singular n-chains on X with coefficients in G, and those of $S^n(X;G)$ are called singular n-cochains on X with coefficients in G.

Thus $S_*(X;G)$ becomes a chain complex of abelian groups with the boundary operator $\partial\colon S_n(X;G) \to S_{n-1}(X;G)$ defined by $\partial(g \otimes \sigma) = \sum_{i=0}^{n} (-1)^i g \otimes (\sigma \circ d_i)$ for every $\sigma\colon \Delta^n \to X$ and all $g \in G$. The homology groups of X with coefficients in G are defined as

$$H_n(X;G) = H_n(S_*(X;G)).$$

Similarly, the cohomology groups of X with coefficients in G are defined as

$$H^n(X;G) = H^n(S^*(X;G)),$$

where $S^*(X;G)$ is viewed as a cochain complex of abelian groups equipped with the coboundary operator $\delta\colon S^n(X;G) \to S^{n+1}(X;G)$ defined as $\delta\varphi = \varphi \circ \partial$ for every $\varphi\colon S_n(X) \to G$.

Kronecker Pairing

For an abelian group G and every n, there is a canonical homomorphism

$$S^n(X;G) \otimes S_n(X) \longrightarrow G$$

sending $\varphi \otimes c$ to $\varphi(c)$. This homomorphism will be denoted by $\langle -, - \rangle$, so that $\langle \varphi, c \rangle = \varphi(c)$. This notation emphasizes the fact that the boundary operator ∂ and the coboundary operator δ are adjoint:

$$\langle \delta\varphi, c \rangle = \langle \varphi, \partial c \rangle \tag{1}$$

for all $\varphi \in S^{n-1}(X;G)$ and $c \in S_n(X)$.

For a continuous map $f\colon X \to Y$, the induced homomorphisms $f_\flat$ and $f^\sharp$ on $S_*(X)$ and $S^*(X;G)$, respectively, are also adjoint:

$$\langle f^\sharp(\varphi), c \rangle = \langle \varphi, f_\flat(c) \rangle. \tag{2}$$

From (1) we obtain, for each n, a homomorphism

$$H^n(X;G) \otimes H_n(X) \longrightarrow G, \tag{3}$$

called *Kronecker pairing* or *Kronecker product*. By (2), for every continuous map $f\colon X \to Y$ we have

$$\langle f^*(b), a \rangle = \langle b, f_*(a) \rangle \tag{4}$$

for every $b \in H^n(Y;G)$ and every $a \in H_n(X)$.

Universal Coefficient Formulas

We may consider the adjoint homomorphism to (3),

$$\theta\colon H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G), \tag{5}$$

which leads to the first form of the *Universal Coefficient Theorem*: for every $n \geq 0$ and every abelian group G, there is a natural exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0, \tag{6}$$

which splits, although the splitting is not natural. Consequently,

$$\boxed{H^n(X;G) \cong \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G)}$$

for every space X, every abelian group G, and all n.

The second form of the Universal Coefficient Theorem states the existence of a natural exact sequence for every $n \geq 0$ and every abelian group G:

$$0 \longrightarrow H_n(X) \otimes G \longrightarrow H_n(X;G) \longrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0, \tag{7}$$

which splits, yet the splitting is not natural either. Hence

$$\boxed{H_n(X;G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}(H_{n-1}(X), G)}$$

for every space X, every abelian group G, and all n.

These exact sequences (6) and (7) are useful for computing homology and cohomology with arbitrary coefficients starting from the homology groups with coefficients in $\mathbb{Z}$. It follows, for example, that if X is a space such that $H_n(X) = 0$ for $n \geq 1$, then $H_n(X;G) = 0$ and $H^n(X;G) = 0$ for $n \geq 1$ and all abelian groups G.

Kronecker Duality

Suppose that K is a field. Then for every space X we may view $S_*(X;K)$ as a chain complex of K-vector spaces. Therefore, the homology groups $H_n(X;K)$ and the cohomology groups $H^n(X;K)$ acquire canonical K-vector space structures.

For each K-vector space V, observe that

$$S_n(X;V) = V \otimes S_n(X) = V \otimes_K K \otimes S_n(X) = V \otimes_K S_n(X;K),$$

and similarly

$$S^n(X;V) = \mathrm{Hom}(S_n(X), V) = \mathrm{Hom}_K(K \otimes S_n(X), V) = \mathrm{Hom}_K(S_n(X;K), V).$$

Hence the Kronecker pairing can be written down as

$$H^n(X;V) \otimes_K H_n(X;K) \longrightarrow V$$

for every K-vector space V, and, in particular,

$$H^n(X;K) \otimes_K H_n(X;K) \longrightarrow K.$$

Since all K-vector spaces are free over K, we have

$$\mathrm{Tor}_1^K(H_{n-1}(X;K), V) = 0, \qquad \mathrm{Ext}^1_K(H_{n-1}(X;K), V) = 0$$

for each K-vector space V and every n. Therefore we obtain, similarly as in the Universal Coefficient Theorem,

$$\boxed{H^n(X;K) \cong \mathrm{Hom}_K(H_n(X;K), K).}$$

This tells us precisely that $H^n(X;K)$ is the dual vector space to $H_n(X;K)$ if K is a field. This form of duality is called *Kronecker duality*.

It follows that, if $H_n(X;K)$ has finite dimension over K, then

$$\dim_K H^n(X;K) = \dim_K H_n(X;K).$$

Exercises

20. Prove that, for every space X and every abelian group G, the zeroth cohomology group $H^0(X;G)$ is isomorphic to a direct product of copies of G indexed by the set $\pi_0(X)$ of path-connected components of X.

21. Prove that $H^1(X)$ is torsion-free for all spaces X.

22. Prove that, if a map $f\colon X \to Y$ induces an isomorphism $H_n(X) \cong H_n(Y)$ for each n, then f also induces isomorphisms $H_n(X;G) \cong H_n(Y;G)$ and $H^n(Y;G) \cong H^n(X;G)$ for all n and every abelian group G.

23. Find the singular homology groups with coefficients in $\mathbb{Z}$, $\mathbb{Z}/2$ and $\mathbb{Q}$ of all compact connected surfaces.

24. Find the singular homology groups with coefficients in $\mathbb{Z}$, $\mathbb{Z}/2$ and $\mathbb{Q}$ of the real projective spaces $\mathbb{R}P^n$ and the complex projective spaces $\mathbb{C}P^n$ for all n.

25. Compute the cohomology groups with coefficients in $\mathbb{Z}$, $\mathbb{Z}/2$ and $\mathbb{Q}$ of the following spaces:

    (i) The sphere S^n for every $n \geq 1$.
    (ii) The n-torus $S^1 \times \cdots \times S^1$ for every $n \geq 1$.
    (iii) All compact connected surfaces.
    (iv) The real projective spaces $\mathbb{R}P^n$ for every $n \geq 1$.
    (v) The complex projective spaces $\mathbb{C}P^n$ for every $n \geq 1$.
    (vi) A product of two spheres $S^n \times S^m$ for any values of n and m.