Homology of Cell Complexes

Cell Complexes

Let us denote by E^n the closed n-ball, whose boundary is the sphere S^{n-1}. A *cell complex* (or *CW-complex*) is a topological space X equipped with a filtration by subspaces

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots \tag{1}$$

where $X^{(0)}$ is discrete (i.e., has the discrete topology), such that $X = \cup_{n=0}^{\infty} X^{(n)}$ as a topological space, and, for all n, the space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching a set of n-cells by means of a family of maps $\{\varphi_j\colon S^{n-1} \to X^{(n-1)}\}_{j\in J_n}$. Here J_n is any set of indices and $X^{(n)}$ is therefore a quotient of the disjoint union $X^{(n-1)} \coprod_{j\in J_n} E^n$ where we identify, for each point $x \in S^{n-1}$, the image $\varphi_j(x) \in X^{(n-1)}$ with the point x in the j-th copy of E^n. It is customary to denote

$$X^{(n)} = X^{(n-1)} \cup_{\{\varphi_j\}} \{e_j^n\}$$

and view each e_j^n as an "open n-cell". The space $X^{(n)}$ is called the *n-skeleton* of X, and the filtration (1) together with the attaching maps $\{\varphi_j\}_{j\in J_n}$ for all n is called a *cell decomposition* or *CW-decomposition* of X. Thus a topological space may admit many distinct cell decompositions (or none).

Cellular Chain Complexes

Let X be a cell complex with n-skeleton $X^{(n)}$ and attaching maps $\{\varphi_j\}_{j\in J_n}$ for all n. We define

$$\boxed{C_n(X) = H_n\big(X^{(n)}, X^{(n-1)}\big)}$$

for each n. Note that, since $X^{(n-1)}$ is a closed subspace of $X^{(n)}$ which is a strong deformation retract of an open neighbourhood, we have

$$C_n(X) \cong H_n\big(X^{(n)}/X^{(n-1)}\big)$$

for all n. Moreover, $X^{(n)}/X^{(n-1)} \cong \vee_{j\in J_n} S^n$. Therefore $C_n(X)$ is a free abelian group with a free generator for each $j \in J_n$, or, in other words, for each n-cell of X. Thus we may write

$$C_n(X) \cong \oplus_{j\in J_n} \mathbb{Z}e_j^n.$$

We next show that $C_*(X)$ is in fact a chain complex whose homology groups are isomorphic to the homology groups of X. For this, we need to define a boundary operator $\partial_n\colon C_n(X) \longrightarrow C_{n-1}(X)$ for each n. We pick $\partial_n = p_{n-1} \circ \Delta_n$, where

$$\Delta_n\colon H_n\big(X^{(n)}, X^{(n-1)}\big) \longrightarrow H_{n-1}\big(X^{(n-1)}\big)$$

is the connecting homomorphism from the homology long exact sequence of the pair $\big(X^{(n)}, X^{(n-1)}\big)$, and

$$p_{n-1}\colon H_{n-1}\big(X^{(n-1)}\big) \longrightarrow H_{n-1}\big(X^{(n-1)}, X^{(n-2)}\big)$$

comes from the homology long exact sequence of the pair $\big(X^{(n-1)}, X^{(n-2)}\big)$. Hence,

$$\partial_{n-1} \circ \partial_n = p_{n-2} \circ \Delta_{n-1} \circ p_{n-1} \circ \Delta_n = 0,$$

since $\Delta_{n-1} \circ p_{n-1} = 0$, as they are consecutive arrows in the same exact sequence.

Furthermore, observe that p_{n-1} is injective because $H_{n-1}\big(X^{(n-2)}\big) = 0$, as one shows inductively. This fact yields

$$\begin{aligned} H_n(C_*(X)) &= \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1} = \operatorname{Ker} \Delta_n / \operatorname{Im} \partial_{n+1} \\ &= \operatorname{Im} p_n / \operatorname{Im} \partial_{n+1} \cong H_n\big(X^{(n)}\big) / \operatorname{Im} \Delta_{n+1} \cong H_n\big(X^{(n)}\big) / \operatorname{Ker} i_n \\ &\cong \operatorname{Im} i_n \cong H_n\big(X^{(n+1)}\big) \cong H_n(X), \end{aligned}$$

where $i_n \colon H_n\big(X^{(n)}\big) \to H_n\big(X^{(n+1)}\big)$ is induced by the inclusion. Note that i_n is surjective since $H_n\big(X^{(n+1)}, X^{(n)}\big) = 0$, and the isomorphism $H_n\big(X^{(n+1)}\big) \cong H_n(X)$ is shown by applying the Mayer–Vietoris exact sequence to the k-cells with $k > n+1$.

Incidence Numbers

Let us obtain a more useful description of the boundary operators ∂_n of the cellular chain complex. For a cell complex X with n-skeleton $X^{(n)}$ and attaching maps $\{\varphi_j\}_{j \in J_n}$, we may view ∂_n as a group homomorphism between free abelian groups:

$$\partial_n \colon \ \oplus_{j \in J_n} \ \mathbb{Z} e_j^n \longrightarrow \oplus_{i \in J_{n-1}} \ \mathbb{Z} e_i^{n-1}.$$

Hence ∂_n is fully determined by an array of integers, which we call *incidence numbers* and denote as follows:

$$\partial_n e_j^n = \sum_{i \in J_{n-1}} [e_i^{n-1} : e_j^n] \, e_i^{n-1}.$$

Observe that $[e_i^{n-1} : e_j^n]$ is the degree of the map $\phi_i^j \colon S^{n-1} \to S^{n-1}$ obtained by composing $\varphi_j \colon S^{n-1} \to X^{(n-1)}$ with the collapse $X^{(n-1)} \to X^{(n-1)}/(X^{(n-1)} \smallsetminus e_i^{n-1})$ and with the canonical homeomorphism of the latter with S^{n-1}.

Simplicial Homology

Suppose that $|K|$ is the geometric realization of an abstract simplicial complex K. Then $|K|$ admits a cell decomposition where the n-cells correspond to the n-faces of K, and the cellular chain complex $C_*(|K|)$ is isomorphic to the simplicial chain complex $C_*(K)$. This shows that

$$H_n(|K|) \cong H_n(K)$$

for all n, where the group on the left means singular homology while the group on the right means simplicial homology.

Exercises

26. Prove that, for every abstract simplicial complex K, the boundary operator of the cellular chain complex $C_*(|K|)$ coincides with the one of the simplicial chain complex $C_*(K)$.

27. Find the homology groups of the space obtained by attaching an $(n+1)$-cell to a sphere S^n by means of a map $f \colon S^n \to S^n$ of degree $k \geq 2$.

28. Prove that for every family $\{A_n\}_{n \geq 1}$ of finitely generated abelian groups there exists a connected space X such that $H_n(X) \cong A_n$ for all n.