Cohomology Products

Cup Product

If X and Y are topological spaces, the Eilenberg–Zilber Theorem states the existence of a natural homotopy equivalence Φ of singular chain complexes

$$S_*(X \times Y) \simeq S_*(X) \otimes S_*(Y).$$

If $\alpha \in S^p(X; G)$ and $\beta \in S^q(Y; G')$ are cochains, where G and G' are abelian groups, we may consider the composition

$$S_{p+q}(X \times Y) \xrightarrow{\Phi} (S_*(X) \otimes S_*(Y))_{p+q} \xrightarrow{\alpha\otimes\beta} G \otimes G',$$

where $\alpha \otimes \beta$ is meant to be zero on all the summands except for $S_p(X) \otimes S_q(Y)$.

This defines a natural homomorphism

$$S^p(X; G) \otimes S^q(Y; G') \xrightarrow{\times} S^{p+q}(X \times Y; G \otimes G'),$$

which will be denoted with a multiplication symbol. The effect of the coboundary operator is given by

$$\delta(\alpha \times \beta) = (\delta\alpha) \times \beta + (-1)^p \, \alpha \times (\delta\beta).$$

This expression allows us to factor this multiplication to cohomology. The resulting operation is known as *external cohomology product*:

$$H^p(X; G) \otimes H^q(Y; G') \xrightarrow{\times} H^{p+q}(X \times Y; G \otimes G').$$

If we now set $X = Y$ and choose $G = G' = R$ (a ring), and consider the *diagonal map* $d\colon X \to X \times X$, $d(x) = (x, x)$, then the composite

$$H^p(X; R)\otimes H^q(X; R) \xrightarrow{\times} H^{p+q}(X\times X; R\otimes R) \xrightarrow{d^*} H^{p+q}(X; R\otimes R) \xrightarrow{\mu_*} H^{p+q}(X; R),$$

where $\mu\colon R \otimes R \to R$ is the multiplication of R, yields an internal product that is called *cup product* and denoted with $\smile$. Thus, if φ and ψ are any two cocycles, the product of the corresponding cohomology classes $[\varphi]$, $[\psi]$ is given by

$$\boxed{[\varphi] \smile [\psi] = [\mu \circ (\varphi \otimes \psi) \circ \Phi \circ d_\flat].} \tag{1}$$

As a further consequence of the Acyclic Model Theorem, all natural chain maps $S_*(X) \to S_*(X) \otimes S_*(X)$ such that $x \mapsto x \otimes x$ for all $x \in S_0(X)$ are homotopic. For this reason, $\Phi \circ d_\flat$ is homotopic to the chain map $\tau\colon S_*(X) \to S_*(X) \otimes S_*(X)$ defined as follows and called *Alexander–Whitney diagonal approximation*:

$$\tau(\sigma) = \sum_{i+j=n} \sigma^i \otimes \sigma_j$$

where

$$\sigma^i(t_0, \dots, t_i) = \sigma(t_0, \dots, t_i, 0, \dots, 0),$$
$$\sigma_j(t_0, \dots, t_j) = \sigma(0, \dots, 0, t_0, \dots, t_j)$$

are called *front i-th face* and *back j-th face* of σ, respectively, for each $\sigma \colon \Delta_n \to X$.

Since $\tau \simeq \Phi \circ d_\flat$, the cup product can also be expressed as

$$[\varphi] \smile [\psi] = [\mu \circ (\varphi \otimes \psi) \circ \tau].$$

Thus, we may define $\varphi \smile \psi$ as the cocycle which takes the following value on each singular simplex $\sigma \colon \Delta_{p+q} \to X$:

$$\boxed{(\varphi \smile \psi)(\sigma) = \varphi(\sigma^p)\, \psi(\sigma_q),}$$

and this definition agrees with (1) after passing to cohomology.

In summary, $H^*(X; R)$ is a *graded ring* for every space X and every ring R. Moreover, given any map $f \colon X \to Y$, the induced arrow $f^* \colon H^*(Y; R) \to H^*(X; R)$ is a homomorphism of graded rings, that is, the cup product commutes with induced homomorphisms in cohomology.

Anti-commutativity

In general, $\varphi \smile \psi$ and $\psi \smile \varphi$ are distinct cocycles. However, in cohomology,

$$\boxed{[\varphi] \smile [\psi] = (-1)^{pq}\, [\psi] \smile [\varphi],} \tag{2}$$

where p is the degree of φ and q is the degree of ψ. This can be inferred from the Acyclic Model Theorem, as follows. For spaces X and Y, let $T \colon X \times Y \to Y \times X$ be the twist map $T(x, y) = (y, x)$. Choose, as above, a natural homotopy equivalence $\Phi \colon S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$. If we define $F(\sigma \otimes \eta) = (-1)^{pq}(\eta \otimes \sigma)$, then F is a natural chain map, so the Acyclic Model Theorem implies that $F \circ \Phi \simeq \Phi \circ T_\flat$, since they agree on 0-simplices. Since $(\varphi \otimes \psi) \circ F = (-1)^{pq}\, \psi \otimes \varphi$ and $T \circ d = d$, this fulfills our goal, by taking $Y = X$.

We say that the ring $H^*(X; R)$ is *commutative in the graded sense* meaning that, in fact, it is anti-commutative by (2). An important consequence is that, if α is a cohomology class of odd degree, then necessarily $\alpha \smile \alpha = 0$ (unless the ring R has exponent 2), since $\alpha \smile \alpha = -(\alpha \smile \alpha)$ by reversing the order of the factors.

Cap Product

The *cap product* takes the following form, for a space X and a ring R:

$$H^p(X; R) \otimes H_n(X; R) \overset{\frown}{\longrightarrow} H_{n-p}(X; R),$$

for $n \geq 0$ and $0 \leq p \leq n$. It is defined by factoring the adjoint of the homomorphism

$$S^p(X; R) \longrightarrow \operatorname{Hom}(S_n(X; R), S_{n-p}(X; R))$$

defined as follows: Given a cocycle $\varphi \in S^p(X; R)$, send each cycle $c \in S_n(X; R)$ to

$$\varphi \frown c = ((\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau))(c).$$

Here τ is the Alexander–Whitney diagonal approximation, and $\varphi \otimes \mathrm{id}$ is to be understood as defined over $S_*(X) \otimes S_*(X)$ taking the 0 value on $S_i(X) \otimes S_j(X)$ unless $i = p$ and $j = q$.

Hence, if we remind the definition of τ, we can define, more explicitly,

$$\boxed{\varphi \frown \sigma = \varphi(\sigma^p)\, \sigma_{n-p}}$$

for each $\sigma\colon \Delta_n \to X$. This definition passes to cohomology, as inferred from the formula

$$\partial(\varphi \frown \sigma) = (-1)^p\, (\varphi \frown \partial c) + (-1)^{p+1}(\delta\varphi \frown c)$$

for the boundary operator.

The following adjunction between the cup product and the cap product is important:

$$\boxed{\langle \varphi \smile \psi, \sigma\rangle = \langle \psi, \varphi \frown \sigma\rangle.}$$

This relation follows directly from the definitions. One also infers directly from the definitions that, in the special case $p = n$, the cap product

$$H^n(X; R) \otimes H_n(X; R) \longrightarrow H_0(X; R)$$

is just the Kronecker pairing if X is path-connected (by identifying $H_0(X; R) \cong R$ as usual).

Exercises

29. Let R be any ring with 1. Prove that, for every space X, the 0-cocycle that sends all points of X to 1 is the unit element of the cohomology ring $H^*(X; R)$.

30. Prove that, given any map $f\colon X \to Y$ of spaces and given a ring R with 1, the induced R-module homomorphism $f^*\colon H^*(Y; R) \to H^*(X; R)$ is in fact a homomorphism of R-algebras.

31. Denote by r and s the retractions of the one-point union $X \vee Y$ onto X and Y respectively, where X and Y are any two pointed spaces. Prove that $r^*\alpha \smile s^*\beta = 0$ for all $\alpha \in H^p(X)$ and $\beta \in H^q(Y)$ if $p \geq 1$ and $q \geq 1$, and infer from this fact that $\tilde{H}^*(X \vee Y) \cong \tilde{H}^*(X) \oplus \tilde{H}^*(Y)$ as graded rings.

32. Compute the cohomology ring of a torus $S^1 \times \cdots \times S^1$ of any dimension.

33. Prove that $H^*(S^n \times S^m) \cong H^*(S^n \vee S^m \vee S^{n+m})$ as graded abelian groups for all n and m, but not as rings.

34. Prove that, if a space X can be written as $X = A \cup B$ where A and B are contractible open subspaces, then $\alpha \smile \beta = 0$ for all cohomology classes α and β of nonzero degree. Deduce from this fact that $S^n \times S^m$ is not the union of any two contractible open subspaces if $n \geq 1$ and $m \geq 1$.

35. The *suspension* ΣX of a nonempty topological space X is the quotient of $X \times [0,1]$ by the equivalence relation which collapses $X \times \{0\}$ onto a point and $X \times \{1\}$ onto another point.

    (a) Prove that $H_n(\Sigma X) \cong H_{n-1}(X)$ and $H^n(\Sigma X) \cong H^{n-1}(X)$ for $n \geq 2$.

    (c) Prove that $\alpha \smile \beta = 0$ for all cohomology classes α and β of nonzero degree in $H^*(\Sigma X)$ for every connected space X.

36. Prove that the cup product and the cap product are adjoint, meaning that

$$\langle \varphi \smile \psi, c \rangle = \langle \psi, \varphi \frown c \rangle,$$

where $\langle -, - \rangle$ is the Kronecker pairing.

37. Prove that $\psi \frown (\varphi \frown c) = (\varphi \smile \psi) \frown c$ for all cochains φ and ψ and every chain c of suitable degree.