Excision and Mayer–Vietoris Sequence

Excision Theorem

Suppose given a space X and two subspaces $U \subseteq A \subseteq X$ such that the closure of U is contained in the interior of A. Then the inclusion of $X \setminus U$ into X induces isomorphisms

$$H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$$

for all n, and similarly for cohomology with arbitrary coefficients.

The Mayer–Vietoris Exact Sequence

A pair of subspaces A, B of a space X form an *excisive couple* if excision yields isomorphisms $H_n(A, A \cap B) \cong H_n(A \cup B, B)$ for all n. For example, if $A \cup B$ is the union of the interiors of A and B, then A and B form an excisive couple. If A and B form an excisive couple, then there is a natural long exact sequence

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(A \cup B) \to H_{n-1}(A \cap B) \to \cdots$$

where the homomorphism $H_n(A \cap B) \to H_n(A) \oplus H_n(B)$ is induced by the inclusions, and $H_n(A) \oplus H_n(B) \to H_n(A \cup B)$ is the difference of the homomorphisms induced by the corresponding inclusions. Similarly, there is a natural long exact sequence for cohomology

$$\cdots \to H^n(A \cup B) \to H^n(A) \oplus H^n(B) \to H^n(A \cap B) \to H^{n+1}(A \cup B) \to \cdots ,$$

also with arbitrary coefficients.

Exercises

38. Prove that, if A and B are subcomplexes of a cell complex X, then A and B form an excisive couple.

39. (a) Prove that, if $V \subseteq U \subseteq X$, then there is a long exact sequence

$$\cdots \to H_n(U, V) \to H_n(X, V) \to H_n(X, U) \to H_{n-1}(U, V) \to \cdots .$$

    (b) Prove that, if A and B form an excisive couple, then there is a long exact sequence

$$\cdots \to H_n(A, A \cap B) \to H_n(X, B) \to H_n(X, A \cup B) \to H_{n-1}(A, A \cap B) \to \cdots .$$

40. A space X with a base point x_0 is called *well pointed* if $\{x_0\}$ is a strong deformation retract of some neighbourhood.

    (a) Prove that, if X and Y are well pointed, then $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for $n \geq 1$, where $\vee$ denotes wedge sum (one-point union).

    (b) Prove that the same result holds for infinitely many wedge summands $\vee_{j \in J} X_j$.

41. Prove that, if X is any space and A is a closed subspace which is a strong deformation retract of some neighbourhood, then $H_n(X, A) \cong \tilde{H}_n(X/A)$ for all n.

42. Prove that, if a space Y is obtained from another space X by attaching an n-cell, where $n \geq 1$, then the inclusion $X \subset Y$ induces

    (i) isomorphisms $H_k(X) \cong H_k(Y)$ for $k < n - 1$ and $k > n$;

    (ii) an epimorphism $H_{n-1}(X) \to H_{n-1}(Y)$;

    (iii) a monomorphism $H_n(X) \to H_n(Y)$.

    Prove that, consequently, if X is a cell complex then $H_k(X^{(m)}) \cong H_k(X)$ for $m > k$.

43. Find the homology groups of the connected sum $M_1 \# M_2$ of two compact connected surfaces in terms of the homology groups of M_1 and M_2.

44. Prove that, if M is an n-dimensional topological manifold, where $n \geq 2$, then

$$H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$$

for every point $x \in M$, and moreover $H_k(M, M \setminus \{x\}) = 0$ if $k \neq n$. Prove that the same result holds for homology with coefficients in any abelian group G.