Orientations in Topological Manifolds

Local Orientations

Let M be a topological manifold of dimension n and let R be a commutative ring with 1. A *local R-orientation* at a point $x \in M$ is a generator of $H_n(M, M \smallsetminus \{x\}; R)$ viewed as a free R-module of rank 1.

An *R-orientation class* on a subspace $U \subseteq M$ is a class $\alpha \in H_n(M, M \smallsetminus U; R)$ such that, for every $x \in U$, the restriction homomorphism

$$i_x^U \colon H_n(M, M \smallsetminus U; R) \longrightarrow H_n(M, M \smallsetminus \{x\}; R)$$

sends α to a local R-orientation at x.

Orientability

An *R-oriented chart* in an n-dimensional manifold M is a pair (U, α) where U is an open subpace of M such that $U \cong \mathbb{R}^n$ and α is an R-orientation class on U.

A topological manifold M is *R-orientable* if there is a set $\{(U_j, \alpha_j)\}_{j \in J}$ of R-oriented charts such that $\cup_{j \in J} U_j = M$ and such that, whenever $U_j \cap U_k \neq \emptyset$, the classes α_j and α_k induce the same local orientation at each point of $U_j \cap U_k$. Such a set of charts is called an *R-oriented atlas*. An *R-orientation* on M is an equivalence class of R-oriented atlases, where two such atlases are equivalent if their union is also an R-oriented atlas.

If $R = \mathbb{Z}$, then it will be omitted from the notation. Thus an *orientable manifold* is a $\mathbb{Z}$-orientable manifold.

Exercises

45. (a) Prove that, if M is $\mathbb{Z}$-orientable, then it is R-orientable for every ring R.
    (b) Prove that all manifolds are $\mathbb{Z}/2$-orientable.

46. Prove that, if a connected manifold M is $\mathbb{Z}$-orientable, then it admits precisely two distinct orientations. More generally, if M is connected and R-orientable, then there is a bijection between the orientations on M and the set R^* of invertible elements in the ring R.

47. Prove that for every point x of a topological manifold M and every ring R there is an R-oriented chart (U, α) in M with $x \in U$.

48. Prove that every submanifold of an R-orientable manifold is R-orientable.

49. Let M be a connected n-dimensional manifold. Prove the following facts:
    (i) $H_k(M) = 0$ if $k > n$.
    (ii) If M is compact and orientable, then $H_n(M) \cong \mathbb{Z}$.
    (iii) If M is noncompact or nonorientable, then $H_n(M) = 0$.

50. (a) Prove that the real projective spaces $\mathbb{R}P^n$ are orientable if and only if n is odd.
    (b) Prove that the complex projective spaces $\mathbb{C}P^n$ are orientable for all n.

51. Prove that all simply-connected manifolds are orientable.