Endomorphisms of Exotic Models

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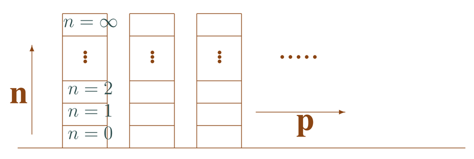
Introduction

The stable homotopy category $\mathrm{Ho}(\mathcal{S})$ is a big and complex category. Thus it becomes natural to break up its p-local parts $\mathrm{Ho}(\mathcal{S}_{(p)})$ into smaller (atomic pieces) given by the *chromatic localisations* $\mathrm{Ho}(L_n\mathcal{S})$, $n \in \mathbb{N}$.

Visualising $\mathrm{Ho}(\mathcal{S})$ *in relation to* $\mathrm{Ho}(L_n\mathcal{S})$*:*



The "ground floor" $\mathrm{Ho}(L_0\mathcal{S})$ is given by rational homotopy theory. The first floor $\mathrm{Ho}(L_1\mathcal{S})$ is governed by p-local topological K-theory which is related to vector bundles.

Schwede showed in [7] that the triangulated structure of $\mathrm{Ho}(\mathcal{S})$ determines the entire higher homotopy information of spectra. In other words, the stable homotopy category is *rigid*. This is fascinating as such examples of rigidity are usually hard to find. A natural question to follow is whether the atomic building blocks $\mathrm{Ho}(L_n\mathcal{S})$ are rigid, too. Franke showed in [4] that for $n^2 + n < 2p - 2$ (in particular for $n = 1$ and p odd) this is actually not the case by constructing an algebraic counterexample. The second author showed in [5] that in the case of $n = 1$ and $p = 2$, the K-local stable homotopy category $\mathrm{Ho}(L_1\mathcal{S})$ is indeed rigid.

Franke's model is *algebraic*, which means that it is model enriched over the model category of chain complexes. Therefore it makes sense to direct the study of exotic models to algebraic models. For example, is Franke's model the only algebraic model for $\mathrm{Ho}(L_1\mathcal{S})$? Or are all exotic models for $\mathrm{Ho}(L_1\mathcal{S})$ algebraic?

By Morita theory, algebraic model categories are determined by an endomorphism dga with homology and Massey products. To get a grip on those uniqueness questions we have to understand the endomorphism dgas. E.g. if there was a unique endomorphism dga, then there'd be a unique algebraic model. This has partially been answered in [6] but it does not seem feasible to approach this by hand due to the rapidly increasing complexity of the computations.

We are going to look at the endomorphism dga of Franke's exotic models. Its construction consists of a many abstract ingredients. We are going to carefully unravel those in order to arrive at the $\mathbb{Z}_p$-module structure of the dga in question. So we turned a big abstract machinery into concrete numbers, which contributes to the greater picture by allowing for direct calculations in the future.

Algebraic Models

The basic goal is to study the K-local stable homotopy category $\mathrm{Ho}(L_1\mathcal{S})$ at an odd prime p. We will study the existence of algebraic model categories, such that there is an equivalence of triangulated categories

$$\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C}).$$

If $\mathcal{C}$ is an arbitrary stable model category, it can be very hard to understand, or to compare $L_1\mathcal{S}$ with $\mathcal{C}$. The following result [8, Theorem 3.1.1] gives a more concrete way to approach $\mathcal{C}$. Recall that an object $X \in \mathrm{Ho}(\mathcal{C})$ is *compact* if the functor $\mathrm{Ho}(\mathcal{C})(X, -)$ commutes with arbitrary coproducts. X is a *generator* if the full subcategory of $\mathrm{Ho}(\mathcal{C})$ containing X which is closed under coproducts and exact triangles is again $\mathrm{Ho}(\mathcal{C})$ itself. Then we have the following result.

Theorem 1. *[Schwede-Shipley] Let $\mathcal{C}$ be a simplicial proper, stable model category with a compact generator X. Then there exists a chain of simplicial Quillen equivalences between $\mathcal{C}$ and module spectra over the endomorphism ring spectrum of X,*

$$\mathcal{C} \simeq \textit{mod-}\,\mathrm{End}(X).$$

The category $\mathrm{Ho}(L_1\mathcal{S})$ possesses the sphere L_1S^0 as a compact generator. Thus if

$$\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C}).$$

is a triangulated equivalence as above, we can use (a fibrant and cofibrant replacement of) $X = \Phi(L_1S^0)$ as a compact generator for $\mathrm{Ho}(\mathcal{C})$.

For algebraic $\mathcal{C}$ we can say that $\mathcal{C}(X, X)$ is a differential graded algebra (dga) rather than just a graded abelian group, and the endomorphism dga $\mathcal{C}(X, X)$ for $X \cong \Phi(L_1S^0)$ satisfies ([6, Lemma 2.1]):

- $H_*(\mathcal{C}(X, X)) = \mathrm{Ho}(\mathcal{C})(X, X) = \pi_*(L_1S^0)$.
- Under the above, the Massey products of $\mathcal{C}(X, X)$ coincide with the Toda brackets of $\pi_*(L_1S^0)$.

Franke's model and its compact generator

We consider the category $\mathcal{B}$, an abelian category which is equivalent to $E(1)_*E(1)$-comodules that are concentrated in degrees $0 \bmod 2p - 2$. We can think of $E(1)_*E(1)$-comodules as modules over $E(1)_*$ with an action of the Adams operations. Furthermore, the category $\mathcal{B}$ is equipped with self-equivalences

$$T^{j(p-1)} : \mathcal{B} \longrightarrow \mathcal{B} \;\; (j \in \mathbb{Z})$$

which is the identity on the underlying $E(1)_*$-modules but changes the Adams operation Ψ^k by a factor of $k^{j(p-1)}$.

Now we consider *twisted chain complexes* $\mathcal{C}^{2p-2}(\mathcal{B})$ on $\mathcal{B}$. An object of $\mathcal{C}^{2p-2}(\mathcal{B})$ is a cochain complex C^* with $C^i \in \mathcal{B}$ together with an isomorphism

$$\alpha_C : T^{(2p-2)(p-1)}(C^*) \longrightarrow C^*[2p-2] = C^{*+2p-2}.$$

Let $\mathcal{D}^{2p-2}(\mathcal{B})$ be the homotopy category of a model category of $\mathcal{C}^{2p-2}(\mathcal{B})$.

Theorem 2 (Franke)**.** *For $p > 2$ there is an equivalence of triangulated categories*

$$\mathcal{R} : \mathcal{D}^{2p-2}(\mathcal{B}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$$

which satisfies

$$\bigoplus_{i=0}^{2p-3} H^i(C)[-i] \cong E(1)_*(\mathcal{R}(C)).$$

The cochain complex $A^* = \mathcal{R}^{-1}(L_1S^0)$ is $C^i = T^{i(p-1)}(E(1)_*)$ in degrees $i = k(2p-2), k \in \mathbb{Z}$ and 0 in all other degrees. It is a compact generator for $\mathcal{D}^{2p-2}(\mathcal{B})$. Hence, to understand Franke's model we need to study the endomorphism dga of A^*.

The endomorphism dga

We use the standard injective resolution by Adams-Baird-Ravenel [1]

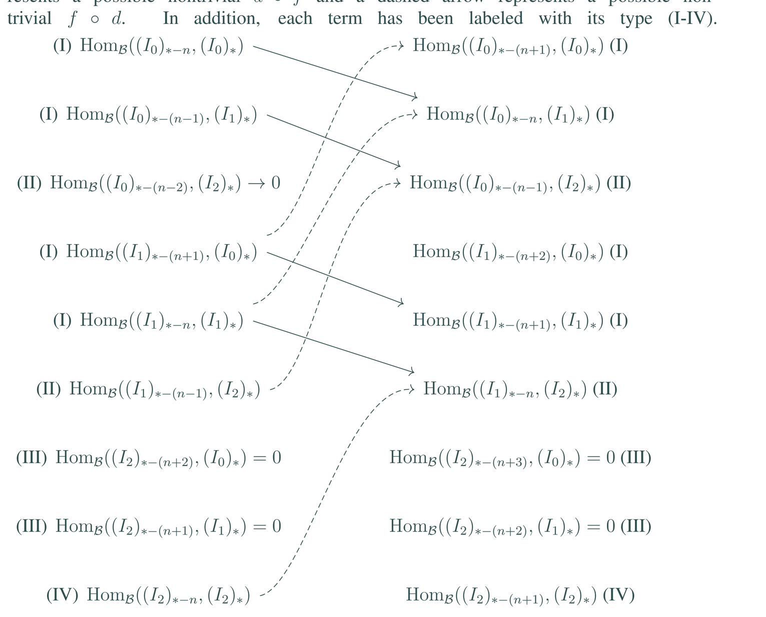
$$0 \longrightarrow E(1)_* \longrightarrow E(1)_*E(1) \xrightarrow{(\Psi^r-1)_*} E(1)_*E(1) \xrightarrow{q} E(1)_* \otimes \mathbb{Q} \longrightarrow 0 \tag{1}$$

where r is a unit of the cyclic group $(\mathbb{Z}/p^2)^\times$, Ψ^r is the r^{th} Adams operation and q is a rational isomorphism in degree 0 and trivial in other degrees. Now the endomorphism cochain complex is

$$C^n := \mathrm{Hom}_{\mathcal{C}^{2p-2}(\mathcal{B})}((A^{fib})^*, (A^{fib})^*)^n = \prod_{n=t-s,i} \mathrm{Hom}_{\mathcal{B}}((I_i)_{*-t}, (I_{i+s})_*)$$

where $I_0 = I_1 = E(1)_*E(1)$ and $I_2 = E(1)_* \otimes \mathbb{Q}$. A differential from C^n to C^{n+1} is of the form $d \circ f + (-1)^{n+1} f \circ d$.

We illustrate its individual parts in the diagram below, where a solid arrow represents a possible nontrivial $d \circ f$ and a dashed arrow represents a possible nontrivial $f \circ d$. In addition, each term has been labeled with its type (I-IV).



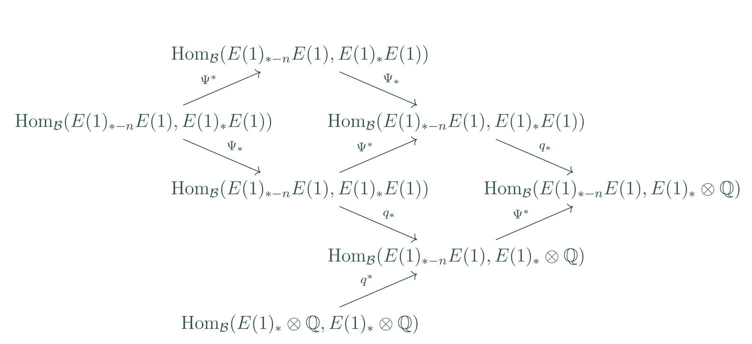
Type (I) $\boxed{\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-t}E(1), E(1)_*E(1)) \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0E(1), \mathbb{Z}_{(p)}(v_1^k)) \quad \text{for } t = (2p-2)k}$

Type (II) $\boxed{\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-t}E(1), E(1)_* \otimes \mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0E(1), \mathbb{Q}\{v_1^k\}) \quad \text{for } t = (2p-2)k}$

Type (III) $\boxed{\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-t} \otimes \mathbb{Q}, E(1)_*E(1)) \cong 0}$

Type (IV) $\boxed{\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-t} \otimes \mathbb{Q}, E(1)_* \otimes \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{for } t = (2p-2)k = 0 \\ 0 & \text{else} \end{cases}}$

$$C^{(2p-2)k-1} \longrightarrow C^{(2p-2)k} \longrightarrow C^{(2p-2)k+1} \longrightarrow C^{(2p-2)k+2}$$



Reinterpretation as Sequences

By [3, Theorem 6.2] we obtain

$$\mathrm{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0E(1), \mathbb{Z}_{(p)}[v_1^k]) \cong E(1)^0E(1) \cong \{\sum_{n\geq 0} a_m\Theta_m(\Psi^r) \mid a_m \in \mathbb{Z}_{(p)}\}$$

This means that we can view the elements of $\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1))$ as sequences of coefficients in p-local integers,

$$\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) \cong \{(a_m)_{m\in\mathbb{N}} \,|a_m \in \mathbb{Z}_{(p)}\} = \mathbb{Z}_{(p)}^{\mathbb{N}}.$$

By the same process, we can consider terms of Type (II) as sequences of coefficients in $\mathbb{Q}$:

$$\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_* \otimes \mathbb{Q}) \cong \{(a_m)_{m\in\mathbb{N}} \,|a_m \in \mathbb{Q}\} = \mathbb{Q}^{\mathbb{N}}.$$

$k = 0$,

$$\Psi^*\langle a_m\rangle = \left\langle \begin{matrix} 0 \\ a_1(r^{s(1)}-1)+a_0 \\ a_2(r^{s(2)}-1)+a_1 \\ \vdots \\ a_m(r^{s(m)}-1)+a_{m-1} \\ \vdots \end{matrix} \right\rangle$$

$$\Psi_* = 0$$

$$q_*\langle a_m\rangle = \langle a_m\rangle$$

$$q^*(x) = \left(\left\langle \begin{matrix} x \\ 0 \\ 0 \\ \vdots \end{matrix} \right\rangle, -x \right)$$

$k \neq 0$,

$$\Psi^*\langle a_m\rangle = \left\langle \begin{matrix} p^{\nu(k)+1}a_0 \\ a_1(r^{s(1)}-1+p^{\nu(k)+1})+a_0 \\ \vdots \\ a_m(r^{s(m)}-1+p^{\nu(k)+1})+a_{m-1} \\ \vdots \end{matrix} \right\rangle$$

$$\Psi_*(v_1^k) = (r^{k(p-1)}-1)v_1^k = p^{\nu(k)+1}v_1^k$$

$$q_*\langle a_m\rangle = 0$$

$$q^* = 0$$

Cohomology calculation for $n = 0$

$$0 \longrightarrow \mathbb{Z}_{(p)}^{\mathbb{N}} \xrightarrow{d^{-1}} \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q} \xrightarrow{d^0} \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q}^{\mathbb{N}} \oplus \mathbb{Q} \xrightarrow{d^1} \mathbb{Q}^{\mathbb{N}} \longrightarrow 0$$

$\boxed{-1}$ $\boxed{0}$ $\boxed{1}$ $\boxed{2}$

$$d^{-1}(\langle a_m\rangle) = \left(\left\langle \begin{matrix} 0 \\ a_1(r^{s(1)}-1)+a_0 \\ \vdots \\ a_m(r^{s(m)}-1)+a_{m-1} \\ \vdots \end{matrix} \right\rangle, \left\langle \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{matrix} \right\rangle, 0 \right) \quad d^0(\langle a_m\rangle, \langle b_m\rangle, x) = \left(\left\langle \begin{matrix} 0 \\ -b_1(r^{s(1)}-1)-b_0 \\ \vdots \\ -b_m(r^{s(m)}-1)-b_{m-1} \\ \vdots \end{matrix} \right\rangle, \left\langle \begin{matrix} b_0-x \\ b_1 \\ \vdots \\ b_m \\ \vdots \end{matrix} \right\rangle, x \right) \quad d^1(\langle a_m\rangle, \langle b_m\rangle, y) = \left\langle \begin{matrix} a_0 \\ a_1+b_1(r^{s(1)}-1)+b_0+y \\ \vdots \\ a_m+b_m(r^{s(m)}-1)+b_{m-1} \\ \vdots \end{matrix} \right\rangle$$

$$\boxed{H^n(C) = \begin{cases} 0 & \text{if } n = -1, \\ \mathbb{Z}_{(p)} & \text{if } n = 0 \\ 0 & \text{if } n = 1, \\ \mathbb{Q}/\mathbb{Z}_{(p)} & \text{if } n = 2 \end{cases}}$$

Cohomology calculation for $n \neq 0$

$$0 \longrightarrow \mathbb{Z}_{(p)}^{\mathbb{N}} \xrightarrow{d^{(2p-2)k-1}} \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Z}_{(p)}^{\mathbb{N}} \xrightarrow{d^{(2p-2)k}} \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q}^{\mathbb{N}} \xrightarrow{d^{(2p-2)k+1}} \mathbb{Q}^{\mathbb{N}} \longrightarrow 0$$

$\boxed{(2p-2)k-1}$ $\boxed{(2p-2)k}$ $\boxed{(2p-2)k+1}$ $\boxed{(2p-2)k+2}$

$$d^{(2p-2)k-1}(\langle a_m\rangle) = \left(\left\langle \begin{matrix} p^\nu a_0 \\ a_1(r^{s(1)}-1+p^\nu)+a_0 \\ \vdots \\ a_m(r^{s(m)}-1+p^\nu)+a_{m-1} \\ \vdots \end{matrix} \right\rangle, \left\langle \begin{matrix} p^\nu a_0 \\ p^\nu a_1 \\ \vdots \\ p^\nu a_m \\ \vdots \end{matrix} \right\rangle \right) \quad d^{(2p-2)k}(\langle a_m\rangle, \langle b_m\rangle) = \left(\left\langle \begin{matrix} p^\nu(a_0-b_0) \\ p^\nu a_1 - b_1(r^{s(1)}-1+p^\nu)-b_0 \\ \vdots \\ p^\nu a_m - b_m(r^{s(m)}-1+p^\nu)-b_{m-1} \\ \vdots \end{matrix} \right\rangle, \left\langle \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{matrix} \right\rangle \right) \quad d^{(2p-2)k+1}(\langle a_m\rangle, \langle b_m\rangle) = \left\langle \begin{matrix} p^\nu b_0 \\ b_1(r^{s(1)}-1+p^\nu)+b_0 \\ \vdots \\ b_m(r^{s(m)}-1+p^\nu)+b_{m-1} \\ \vdots \end{matrix} \right\rangle$$

$$\boxed{H^n(C) = \begin{cases} \mathbb{Z}/p^\nu & \text{if } n = (2p-2)k+1 \\ 0 & \text{else} \end{cases}}$$

Product and Massey Products

The multiplication $C^{-(2p-2)k+1} \otimes C^{(2p-2)k+1} \rightarrow C^2$ induces the canonical multiplication on homology. That is, the multiplication $H^{-(2p-2)k+1}(C) \otimes H^{(2p-2)k+1}(C) \rightarrow H^2(C)$ is given by

$$\begin{aligned} \mathbb{Z}/p^{\nu(k)+1} \otimes \mathbb{Z}/p^{\nu(k)+1} &\longrightarrow \mathbb{Q}/\mathbb{Z}_{(p)} \\ a \otimes b &\longmapsto \frac{a}{p^{\nu(k)+1}}\frac{b}{p^{\nu(k)+1}} \end{aligned}$$

and it is injective.

Take γ_k an element of the cohomology $H^{(2p-2)k+1}(C) \cong \mathbb{Z}/p^{\nu(k)+1}$ such that $p\gamma_k = 0$. Then γ_k satisfy the following Massey product relation:

$$\langle \gamma_i, p, \gamma_j \rangle = \gamma_{i+j}$$

and the indeterminancy of this product is zero.

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