Logarithmic structures in homotopy theory

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Rigidification of homotopy coherent commutative multiplications

Let $\mathcal{I}$ be the category of finite sets, $\underline{m} = \{1, \ldots, m\}$ for $m \geq 0$ with the convention that $\underline{0} = \emptyset$, and injective maps. The ordered concatenation $\sqcup$ makes $\mathcal{I}$ a symmetric strict monoidal category with unit $\underline{0}$ and non-trivial symmetry isomorphisms the shuffle maps.

For $(\mathcal{C}, \otimes, \mathbb{1}_\mathcal{C})$ a symmetric monoidal category, the functor category $\mathcal{C}^\mathcal{I}$ inherits a symmetric monoidal structure: For $M, N \in \mathcal{C}^\mathcal{I}$, the product $M \boxtimes N$ is the left Kan extension of $\mathcal{I} \times \mathcal{I} \xrightarrow{M \times N} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ along $\mathcal{I} \times \mathcal{I} \xrightarrow{\sqcup} \mathcal{I}$. So

$$M \boxtimes N(\underline{m}) = \mathrm{colim}_{\underline{n} \sqcup \underline{l} \to \underline{m}} M(\underline{n}) \otimes N(\underline{l})$$

where the colimit is taken over the comma category $\_ \sqcup \_ \downarrow \underline{m}$.

We consider $\mathcal{C} = \mathcal{S}$, the category of spaces, interpreted either as the category of unpointed simplicial sets or as the category of unpointed compactly generated weak Hausdorff topological spaces. The category of commutative $\mathcal{I}$-spaces, $\mathrm{Com}\mathcal{S}^\mathcal{I}$, is the category of commutative monoids in $(\mathcal{S}^\mathcal{I}, \boxtimes, \mathcal{I}(\underline{0}, \_))$.

The weak equivalences in $\mathcal{S}^\mathcal{I}$ are the maps $M \to N$ such that the induced map $\mathrm{hocolim}_\mathcal{I} M \to \mathrm{hocolim}_\mathcal{I} N$ is a weak equivalence in $\mathcal{S}$. The additional symmetry of $\mathcal{I}$-spaces and the use of a positive model structure ensure that E_∞ spaces have strictly commutative models in $\mathcal{I}$-spaces.

Proposition 1. (Sagave-Schlichtkrull) The operad morphism $E_\infty \to \mathrm{Com}$ induces a chain of Quillen equivalences $\mathrm{Com}\mathcal{S}^\mathcal{I} \simeq E_\infty \mathcal{S}$.

Specializing to the Barratt-Eccles operad $\mathcal{E}$ with rth space $\mathcal{E}(r) = E\Sigma_r$, a commutative $\mathcal{I}$-space is sent to its Bousfield-Kan homotopy colimit.

Example 2. Let A be a positive fibrant commutative symmetric ring spectrum. The commutative $\mathcal{I}$-space $\Omega^\mathcal{I}(A) \colon \underline{m} \mapsto \Omega^m(A_m)$ models the underlying multiplicative monoid of A, $\Omega^\infty_\otimes(A)$.

Example 3. (Schlichtkrull-Solberg) Let $(\mathcal{A}, \otimes, \mathbb{1}_\mathcal{A})$ be a small symmetric strict monoidal category. Let $\Phi(\mathcal{A})(\underline{m})$ be the category with objects m-tupels $(a_1, \ldots, a_m)$ of objects in $\mathcal{A}$ and morphism sets

$$\mathrm{Hom}((a_1, \ldots, a_m), (b_1, \ldots, b_m)) = \mathcal{A}(a_1 \otimes \ldots \otimes a_m, b_1 \otimes \ldots \otimes b_m).$$

Permutation of entries and insertion of the unit object $\mathbb{1}_\mathcal{A}$ makes $\phi(\mathcal{A})$ functorial in $\mathcal{I}$. The symmetric strict monoidal structure of $\mathcal{A}$ makes $B\Phi(\mathcal{A}) \colon \underline{m} \mapsto B\Phi(\mathcal{A})(\underline{m})$ a commutative $\mathcal{I}$-space which models $B\mathcal{A}$.

Let k denote a commutative ring with unit and let $\mathrm{Ch(k)}$ denote the category of unbounded chain complexes. In the algebraic context Richter and Shipley showed that there is a chain of Quillen equivalences $\mathrm{ComCh(k)}^\mathcal{I} \simeq E_\infty \mathrm{Ch(k)}$. For example, the cochains on a topological space X with coefficients in k, $S^*(X, k)$, form an E_∞ algebra in cochain complexes. It would be interesting to find an explicit corresponding commutative $\mathcal{I}$-cochain complex model.

Logarithmic ring spectra

Logarithmic ring spectra generalize the algebraic geometric notion of logarithmic rings. They have been studied by Rognes, Sagave and Schlichtkrull. We need graded notions of multiplicative monoids and units for ring spectra.

Let $\mathcal{J}$ denote Quillen's localization construction $\Sigma^{-1}\Sigma$ on the category of finite sets and bijections Σ. It is symmetric strict monoidal and gives rise to $\mathcal{J}$-spaces, $\mathcal{S}^\mathcal{J}$, and a category of commutative $\mathcal{J}$-spaces, $\mathrm{Com}\mathcal{S}^\mathcal{J}$, as in the case of $\mathcal{I}$.

There is a Quillen adjunction relating commutative $\mathcal{J}$-spaces to commutative symmetric ring spectra, $\mathrm{Com}Sp^\Sigma(\mathcal{S}_*, S^1)$.

$$\mathbb{S}^\mathcal{J} \colon \mathrm{Com}\mathcal{S}^\mathcal{J} \rightleftarrows \mathrm{Com}Sp^\Sigma(\mathcal{S}_*, S^1) \colon \Omega^\mathcal{J}$$

For a $\mathcal{J}$-space M the object $\mathbb{S}^\mathcal{J}[M]$ is defined as the coend of the $(\mathcal{J}^{\mathrm{op}} \times \mathcal{J})$-diagram $F_{m_1}(S^{m_2}) \wedge M(\underline{n_1}, \underline{n_2})_+$, and for a symmetric spectrum A the object $\Omega^\mathcal{J}(A)$ is defined by $\mathrm{Map}_{\mathrm{Sp}^\Sigma(\mathcal{S}_*, S^1)}(F\_(S-), A)$. We find that

$$\mathbb{S}^\mathcal{J}[M]_n = \bigvee_{l \geq 0} S^l \wedge_{\Sigma_l} M(\underline{n}, \underline{l})_+ \quad \text{and} \quad \Omega^\mathcal{J}(A)(\underline{n_1}, \underline{n_2}) = \Omega^{n_2}(A_{n_1}).$$

Definition 4. Let A be a commutative symmetric ring spectrum. A **pre-log structure** on A is a commutative $\mathcal{J}$-space M together with a map $M \xrightarrow{\alpha} \Omega^\mathcal{J}(A)$ in $\mathrm{Com}\mathcal{S}^\mathcal{J}$. We call the triple (A, M, α) a **pre-log ring spectrum**. A map of pre-log ring spectra $(A, M, \alpha) \xrightarrow{(f, f^b)} (B, N, \beta)$ consists of a map $A \xrightarrow{f} B$ in $\mathrm{Com}Sp^\Sigma(\mathcal{S}_*, S^1)$ and a map $M \xrightarrow{f^b} N$ in $\mathrm{Com}\mathcal{S}^\mathcal{J}$ such that $\Omega^\mathcal{J}(f)\alpha = \beta f^b$.

Sagave and Schlichtkrull showed that there is a chain of Quillen equivalences $\mathrm{Com}\mathcal{S}^\mathcal{J} \simeq E_\infty\mathcal{S}/B\mathcal{J}$. This allows us to interpret commutative $\mathcal{J}$-spaces as graded E_∞ spaces. We think of $\Omega^\mathcal{J}(A)$ as the underlying graded multiplicative monoid of A. Assume that A is a positive fibrant symmetric ring spectrum. The units $Gl_1^\mathcal{J}(A)$ of A is the sub-commutative $\mathcal{J}$-space of $\Omega^\mathcal{J}(A)$ consisting of the path components that map to units in $\pi_0(\mathrm{hocolim}_\mathcal{J} \Omega^\mathcal{J}(A))$. The canonical map $Gl_1^\mathcal{J}(A) \twoheadrightarrow \Omega^\mathcal{J}(A)$ corresponds to the inclusion $\pi_*(A)^\times \to \pi_*(A)$.

Definition 5. A pre-log ring spectrum (A, M, α) is a **log ring spectrum** if the map $\tilde{\alpha}$ in the following pullback diagram

$$\begin{array}{ccc} \alpha^{-1}Gl_1^\mathcal{J}(A) & \xrightarrow{\tilde{\alpha}} & Gl_1^\mathcal{J}(A) \\ \downarrow & & \downarrow \iota \\ M & \xrightarrow{\alpha} & \Omega^\mathcal{J}(A) \end{array}$$

is a weak equivalence in $\mathrm{Com}\mathcal{S}^\mathcal{J}$, that is, $\mathrm{hocolim}_\mathcal{J}\tilde{\alpha}$ is a weak equivalence in $E_\infty\mathcal{S}/B\mathcal{J}$.

Example 6. The triple $(A, Gl_1^\mathcal{J}(A), \iota)$ is the trival log ring spectrum.

Example 7. The topological K-theory spectrum ku extends to a logarithmic ring spectrum. The log structure on ku is induced by the connective cover map $ku \xrightarrow{f} KU$ and defined via the following pullback diagram

$$\begin{array}{ccc} f_*Gl_1^\mathcal{J}(KU) & \xrightarrow{\tilde{\iota}} & \Omega^\mathcal{J}(ku) \\ \downarrow & & \downarrow \Omega^\mathcal{J}(f) \\ Gl_1^\mathcal{J}(KU) & \xrightarrow{\iota} & \Omega^\mathcal{J}(KU). \end{array}$$

Postnikov towers

Postnikov towers of spectra can be refined to commutative ring spectra. There is a Postnikov tower for the extra data of a pre-log structure on a given ring spectrum.
Let $j \geq 0$. Let $\mathcal{J}_+$ be the full subcategory of $\mathcal{J}$ with objects $(\underline{m_1}, \underline{m_2})$ such that $|\underline{m_1}| \geq 1$.

Definition 8. Let M be a commutative $\mathcal{J}$-space. A jth **Postnikov section** of M is an object N_j in $\mathrm{Com}\mathcal{S}^\mathcal{J}$ together with a morphism $M \xrightarrow{f} N_j$ in $\mathrm{Com}\mathcal{S}^\mathcal{J}$ such that for $(\underline{m_1}, \underline{m_2}) \in \mathcal{J}_+$

- $\pi_l(N_j(\underline{m_1}, \underline{m_2}), *) = 0$ for $l + m_2 - m_1 \geqslant j + 1$, and
- $\pi_l(M(\underline{m_1}, \underline{m_2}), *) \xrightarrow{\pi_l(f(\underline{m_1}, \underline{m_2}))} \pi_l(N_j(\underline{m_1}, \underline{m_2}), *)$ is an isomorphism for $l + m_2 - m_1 \leq j$.

If N_j together with $M \to N_j$ is a jth Postnikov section of M, the sequence

$$M \longrightarrow N_0 \longleftarrow N_1 \longleftarrow \ldots \longleftarrow N_j \longleftarrow N_{j+1} \longleftarrow \ldots$$

is called a **Postnikov tower** for M.

The following example motivated the above definition in the first place:

Example 9. Let A be a connective and positive fibrant symmetric ring spectrum. If we apply the functor $\Omega^\mathcal{J}$ to a positive fibrant model of a Postnikov tower for A, we obtain a Postnikov tower for $\Omega^\mathcal{J}(A)$.

Definition 10. Let (A, M, α) be a pre-log ring spectrum. A jth **Postnikov section** of (A, M, α) is a pre-log ring spectrum (B_j, N_j, α_j) together with a morphism $(A, M, \alpha) \xrightarrow{(f, f^b)} (B_j, N_j, \alpha_j)$ in pre-log ring spectra such that B_j together with f is a jth Postnikov section of A in commutative symmetric ring spectra and N_j together with f^b is a jth Postnikov section of M in commutative $\mathcal{J}$-spaces.

Further goals are to describe logarithmic k-invariants and to find a suitable notion of logarithmic topological André-Quillen cohomology.

Logarithmic dgas

Richter and Shipley showed that there is a chain of Quillen equivalences between the category of commutative Hk-algebra spectra and the category of E_∞ dgas. The logarithmic ring spectra considered so far either come from logarithmic rings or involve topological K-theory spectra. We aim to develop a notion of logarithmic structures for E_∞ dgas and then aim to find interesting examples for logarithmic ring spectra that arise from logarithmic E_∞ dgas.

To approach this question we investigate Richter and Shipley's chain of Quillen equivalences and find corresponding categories of 'underlying graded multiplicative monoids'. An intermediate category in the chain of Quillen equivalences is the category of commutative symmetric ring spectra in unbounded chain complexes with respect to the one-sphere chain complex $\mathbb{S}^1(k)$, denoted by $\mathrm{ComSp}^\Sigma(\mathrm{Ch(k)}, \mathbb{S}^1(k))$. Similarly to the topological case we have a Quillen adjunction

$$\mathbb{S}^\mathcal{J}_{\mathrm{Ch(k)}} \colon \mathrm{ComCh(k)}^\mathcal{J} \rightleftarrows \mathrm{Com}Sp^\Sigma(\mathrm{Ch(k)}, \mathbb{S}^1(k)) \colon \Omega^\mathcal{J}_{\mathrm{Ch(k)}}$$

where for a $\mathcal{J}$-chain complex M the symmetric spectrum $\mathbb{S}^\mathcal{J}_{\mathrm{Ch(k)}}[M]$ in level n is given by $\bigoplus_{l \geq 0} \mathbb{S}^l(k) \otimes_{k\Sigma_l} M(\underline{n}, \underline{l})$, and for a symmetric spectrum A the $\mathcal{J}$-chain complex $\Omega^\mathcal{J}_{\mathrm{Ch(k)}}(A)$ in level $(\underline{n_1}, \underline{n_2})$ is given by $\mathbb{S}^{-n_2}(k) \otimes A_{n_1}$.

We can use the above Quillen adjunction to define a (pre-) log ring spectrum in chain complexes as in the topological context.

References

- Birgit Richter, Brooke Shipley, *An algebraic model for commutative $H\mathbb{Z}$-algebras*, preprint arXiv: 1411.7238.
- Steffen Sagave, Christian Schlichtkrull, *Diagram spaces and symmetric spectra*, Adv. Math. 231 (2012), no. 3-4, 2116-2193.
- Christian Schlichtkrull, Mirjam Solberg, *Braided injections and double loop spaces*, Trans. Amer. Math. Soc. 368 (2016), no. 10, 7305-7338.

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