

# New infinite loop space operads

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This talk is a report on joint work with:

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and Sarah Yeakel**

## Operads: the very basic idea

- codify operations

$$\mu : A^n \longrightarrow A$$

- study families of 'multiplications'

$$\mu : \mathcal{V}(n) \times A^n \longrightarrow A$$

- and how they fit together for different  $n$
- codify notions 'up to homotopy'

## The fundamental group $\pi_1(X, *)$

$\Omega X = \text{maps}([0, 1], \partial; X, *) = \text{loop space on } (X, *)$

**multiplication** defined by subdividing  $[0, 1]$  into two pieces and running loops at a faster rate;

**associativity** and **inverses** — **up to homotopy!**

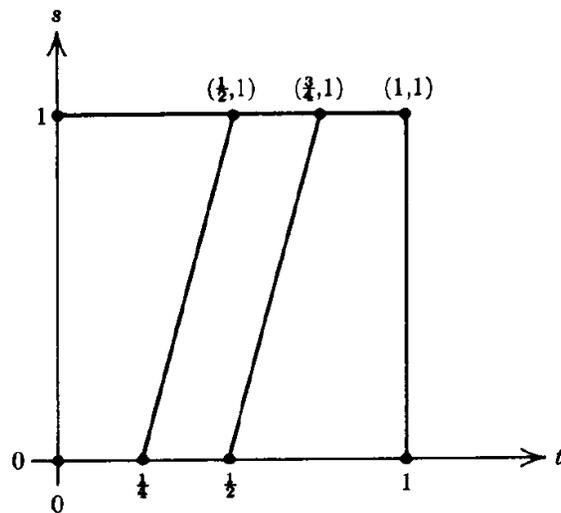


FIGURE 2.1. Proof of associativity.

*From: William S. Massey*

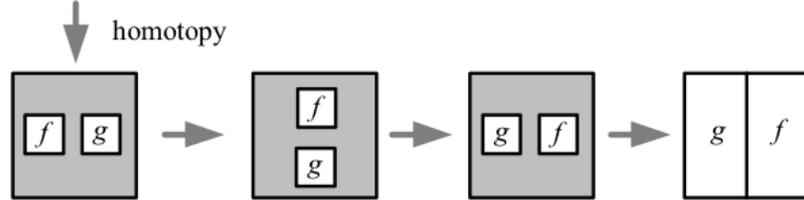
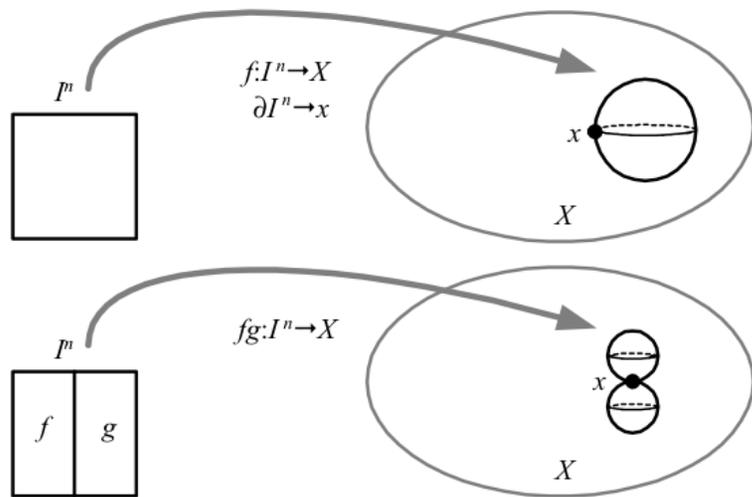
## Higher homotopy groups $\pi_k(X, *)$

$\pi_k(X, *)$  is an abelian group for  $k \geq 2$

$$\begin{aligned}\Omega^k X &= \text{maps}([0, 1]^k, \partial; X, *) \\ &= \Omega(\Omega(\dots \Omega(X, *) \dots)) = k\text{-fold loop space on } (X, *)\end{aligned}$$

commutative – **up to homotopy!**

note: this is true already for  $k = 2$



*From: Adam Marsh*

there is an obvious alternative homotopy;  
 are the homotopies homotopic? yes, for  $\Omega^3(X)$ ;  
 are the homotopies of homotopies homotopic? yes, for  $\Omega^4(X)$ ;  
 ...

$\Omega^\infty$ -spaces = homotopy everything spaces

## Classically (1970s):

- to study operations up to homotopy  
(associative, commutative, ...)
- to give recognition principles for higher loop spaces,  
especially infinite loop spaces

**Recall:**  $\Omega^\infty$ -spaces = homotopy everything spaces  
give rise to generalized (co)homology theories:

$$H_*(-; A), K^*(-), \pi_*^s(-), \Omega_*(-)$$

they have Mayer-Vietoris sequences; hence  
they are computable by divide and conquer techniques!

## Renaissance (1990s):

- to study new structures motivated by mathematical physics (TQFTs and CTFs, string theory, ...)



- moduli spaces and Segal's category  $\mathcal{M}$
- Ginzburg-Kapranov Koszul duality
- Deligne conjecture and Kontsevich formality

## Fusion of classical and renaissance outlook:

Goal: prove that certain operads, such as studied in TQFT and CFT, are infinite loop space operads in the sense that they provide a **compeletely different way** of detecting infinite loop spaces and maps between them.

Main input: **homology stability!**

Homology stability has been a very active research area:

- mapping class groups and related groups
- topological moduli spaces of higher dim. manifolds
- moduli spaces of graphs

## Content

**Part I. Standard definitions and constructions**

**Part II. Operads with homology stability**

**Part III. Main theorem, examples and applications**

**Part IV. Proof**

## PART I

### Operads and their algebras

A topological **operad** is a collection of spaces

$$\mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0}$$

with  $*$   $\in \mathcal{V}(0)$ ,  $1 \in \mathcal{V}(1)$ , a right action of the symmetric group  $\Sigma_n$  on  $\mathcal{V}(n)$  and equivariant structure maps

$$\gamma: \mathcal{V}(k) \times [\mathcal{V}(j_1) \times \dots \times \mathcal{V}(j_k)] \longrightarrow \mathcal{V}(j_1 + \dots + j_k)$$

A  $\mathcal{V}$ -**algebra** is a based space  $(X, *)$  with equivariant structure maps

$$\theta: \mathcal{V}(j) \times X^j \rightarrow X$$

If  $(X, *)$  is a based space, the **free  $\mathcal{V}$ -space on  $X$**  is defined by

$$\mathbb{V}(X) := \coprod_{n \geq 0} (\mathcal{V}(n) \times_{\Sigma_n} X^n) / \sim$$

where  $\sim$  is a base point relation generated by

$$(\sigma_i c; x_1, \dots, x_{n-1}) \sim (c; s_i(x_1, \dots, x_{n-1}))$$

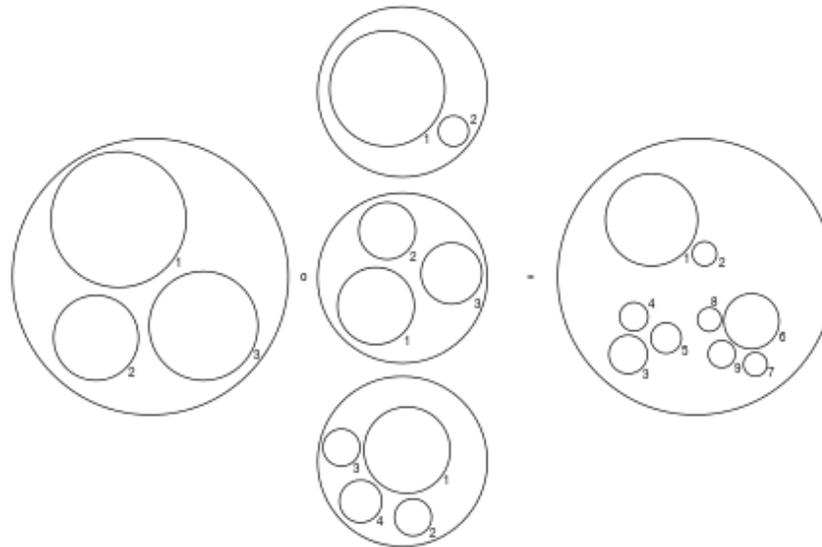
**Example:** note that in general  $\mathcal{V}(0) \neq *$ !  
it defines a non-trivial example of a  $\mathcal{V}$ -algebra

**Morphisms** of operads and algebras - natural definition!

## Classical example:

$\mathcal{C}_n$  with  $\mathcal{C}_n(k) \subset \text{Emb}(\coprod_k D^n, D^n)$

determined by midpoints and radii, and  $\simeq \text{Conf}_k(\mathbb{R}^n)$



$$\gamma : \mathcal{C}_2(3) \times [\mathcal{C}_2(2) \times \mathcal{C}_2(3) \times \mathcal{C}_2(4)] \longrightarrow \mathcal{C}_2(9)$$

Maps of operads:

$$\mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \cdots \rightarrow \mathcal{C}_n \rightarrow \cdots \rightarrow \mathcal{C}_\infty$$

**Example:**  $\Omega^n(X)$  is a  $\mathcal{C}_n$ -algebra

**Recognition Theorem:**

Connected (group-like)  $\mathcal{C}_n$ -algebras are  $\Omega^n$ -spaces.  
More generally, the group completion of an  $\mathcal{C}_n$ -algebra is an  $\Omega^n$ -space.

Stasheff, Broadman-Vogt, May, Barrett-Eccles,  
Milgram, ...

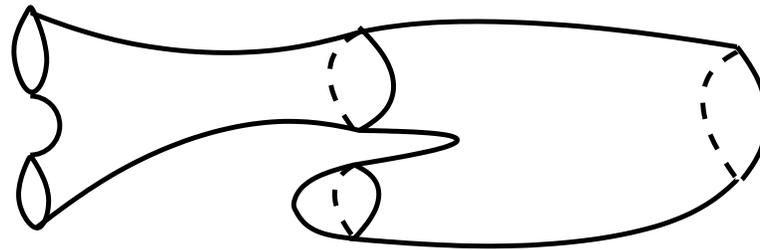


**Rainer Vogt 1942-2015**

## CFT example:

$\mathcal{M}$  with  $\mathcal{M}(n) = \coprod_{g \geq 0} \mathcal{M}_{g,n+1}$

$\mathcal{M}_{g,n+1}$  = **moduli space of Riemann surfaces** of genus  $g$  with  $n + 1$  boundary components



$$\gamma : \mathcal{M}_{0,2+1} \times [\mathcal{M}_{0,2+1} \times \mathcal{M}_{0,0+1}] \longrightarrow \mathcal{M}_{0,2+1}$$

**Example:** a CFT is a symmetric monoidal functor  $\mathcal{F}$  from Segal's category of Riemann surfaces to another symmetric monoidal category  $\mathcal{L}$ ;  
 $\mathcal{F}(S^1)$  is an  $\mathcal{M}$ -algebra when  $\mathcal{L} = \text{spaces}$

## Aside: Group completion

**Algebraic:**  $M \longrightarrow \mathcal{G}(M) =$  Grothendieck group of  $M$

Example:  $\mathbb{N} \longrightarrow \mathcal{G}(\mathbb{N}) = \mathbb{Z}$

**Homotopy theoretic:**

$M \longrightarrow \Omega BM =$  loop space of  $BM$

- $M = G$  a group  $\implies \Omega BG \simeq G$
- $M$  discrete  $\implies \Omega BM \simeq \mathcal{G}(M)$

### Group Completion Theorem:

Let  $M = \bigsqcup_{n \geq 0} M_n$  be a topological monoid such that the multiplication on  $H_*(M)$  is commutative. Then

$$H_*(\Omega BM) = \mathbb{Z} \times \lim_{n \rightarrow \infty} H_*(M_n) = \mathbb{Z} \times H_*(M_\infty)$$

## PART II

### Operads with homology stability OHS

Let  $I$  be a discrete abelian monoid. An  $I$ -**grading** on an operad  $\mathcal{V}$  is a decomposition

$$\mathcal{V}(n) = \coprod_{g \in I} \mathcal{V}_g(n)$$

for each  $n$  so that:

- the basepoint  $*$  lies in  $\mathcal{V}_0(0)$ ,
- the  $\Sigma_n$  action restricts to an action on each  $\mathcal{V}_g(n)$
- the structure maps restrict to maps

$$\gamma: \mathcal{V}_g(k) \times [\mathcal{V}_{g_1}(j_1) \times \dots \times \mathcal{V}_{g_k}(j_k)] \rightarrow \mathcal{V}_{g+g_1+\dots+g_k}(\sum j_i).$$

Assume that for some  $A_\infty$ -operad  $\mathcal{A}$  there is a map of operads

$$\mu : \mathcal{A} \rightarrow \mathcal{V}$$

and the image  $\mu\mathcal{A}(2) \subset \mathcal{V}_0(2)$  is path-connected.

**Definition OHS:**

$\mathcal{V}$  is said to be an operad with **homology stability** if the maps

$$D = \gamma(-; *, \dots, *) : \mathcal{V}_g(n) \rightarrow \mathcal{V}_g(0)$$

are homology isomorphism in degrees  $* < \phi(g)$  where  $\phi$  goes to infinity as  $g$  goes to infinity.

## Classical example:

$$\mu = \text{incl} : \mathcal{C}_2 \rightarrow \mathcal{C}_\infty$$

$\mathcal{C}_\infty(n) \simeq E\Sigma_n \simeq *$  satisfies homology stability trivially

## CFT example:

$$\mu = \text{incl} : \mathcal{C}_2 \rightarrow \mathcal{M}$$

$$\mathcal{M}_g(n) = \mathcal{M}_{g,n+1} \simeq B\text{Diff}(F_{g,n+1}; \partial) \simeq B\Gamma_{g,n+1}$$

[Harer]

$H_*(B\Gamma_{g,n+1})$  is independent of  $g$  and  $n$  for  $g$  large enough

## PART III

**Main Theorem:** [Basterra, Bobkova, Ponto, T., Yeakey]

If  $\mathcal{V}$  is an OHS, group completion

$$\mathcal{G} : \mathcal{V} \text{ - spaces} \longrightarrow \Omega^\infty \text{ - spaces}$$

defines a functor from  $\mathcal{V}$ -spaces to infinite loop spaces  
with a compatible  $\Omega^\infty$ -map

$$\mathcal{G}\mathbb{V}(\ast) \times \mathcal{G}X \rightarrow \mathcal{G}X.$$

## Examples and applications

Non-orientable surfaces:

$$\mathcal{K} \text{ with } \mathcal{K}(k) = \coprod_{g \geq 0} \mathcal{K}_{g,k+1}$$

$$\mathcal{K}_g(k) = \mathcal{K}_{g,k+1} \simeq B\text{Diff}(N_{g,k+1}; \partial) \simeq B\mathcal{N}_{g,k+1}$$

Homology stability: [Wahl]

$H_*(B\mathcal{N}_{g,k+1})$  is independent of  $g, k$  for  $g$  large enough

Spin/Pin surfaces:

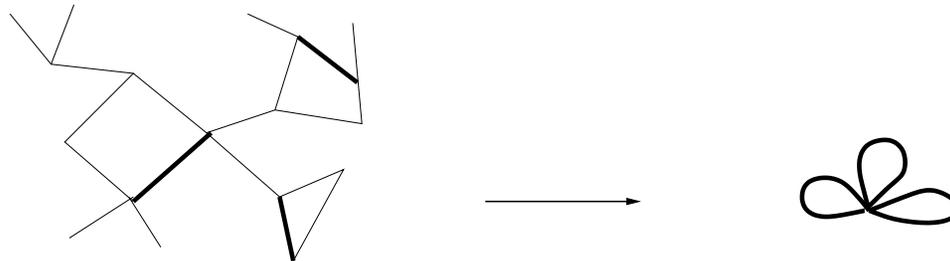
Homology stability: [Harer, Bauer, Randal-Williams]

Moduli space of graphs:

*Graphs* with  $\text{Graphs}(k) = \coprod_{g \geq 0} \text{Graphs}_{g,k+1}$

$\text{Graphs}_{g,k+1}$   
= **Culler-Vogtmann's** Outer space of type  $W_{g,k+1}$   
 $\simeq \text{BHtEq}(W_{g,k+1}; \partial)$

$W_{g,k+1}$  = graph with  $g$  circuits and  $k + 1$  ends)



Homology stability: [**Hatcher-Vogtmann-Wahl**]

$H_*(\text{BHtEq}(W_{g,k+1}; \partial))$  is independent of  $g, k$  for  $g$  large

## Application: constructing $\Omega^\infty$ -maps

the action on  $H_1(N_{g,1}) = \mathbb{F}_2^g$  induces a representation

$$\rho : \mathcal{N}_{g,1} \longrightarrow \mathrm{GL}_g \mathbb{F}_2$$

$\coprod_{g \geq 0} B\mathcal{N}_{g,1} \simeq \mathcal{K}(\ast)$  is the free  $\mathcal{K}$ -algebra on  $\ast$

$\coprod_{g \geq 0} B\mathrm{GL}_g \mathbb{F}_2$  is a  $\mathcal{K}$ -algebra via  $\rho$

$\rho$  is a map of  $\mathcal{K}$ -algebras;

hence induces a map of  $\Omega^\infty$ -spaces:

$$\rho : \mathbb{Z} \times B\mathcal{N}_\infty^+ \simeq \Omega^\infty \mathbf{MTO}(2) \longrightarrow \mathbb{Z} \times B\mathrm{GL}(\mathbb{F}_2)^+ \simeq K(\mathbb{F}_2)$$

## Application: detecting $\Omega^\infty$ -spaces

$\tilde{\mathcal{N}}_{g,1}$  is defined as an extension of  $\mathcal{N}_{g,1}$  by  $H_1(N_{g,1}) = \mathbb{F}_2^g$

$X := \coprod_{g \geq 0} B\tilde{\mathcal{N}}_{g,1}$  is an  $\mathcal{K}$ -algebra

$\mathbb{Z} \times B\tilde{\mathcal{N}}_\infty^+$  is an  $\Omega^\infty$ -space

## PART IV

### Sketch of Proof in 4 Steps

#### Step 1.: Replacement

Let  $\mathcal{V}$  be an OHS. Then the product operad

$$\tilde{\mathcal{V}} := \mathcal{V} \times \mathcal{C}_\infty$$

is an OHS with compatible maps of operads

$$\mathcal{V} \xleftarrow{\pi_1} \tilde{\mathcal{V}} \xrightarrow{\pi} \mathcal{C}_\infty$$

So any  $\mathcal{V}$ -space is a  $\tilde{\mathcal{V}}$ -space.

**W.l.o.g. assume there exists  $\pi : \mathcal{V} \rightarrow \mathcal{C}_\infty$**

## Step 2.: Group completion for free $\mathcal{V}$ -algebras

For any based space  $X$ ,

$$\tau \times \pi: \mathbb{V}(X) \longrightarrow \mathbb{V}(*) \times \mathbb{C}_\infty(X),$$

induces a map of limits of filtration quotients

$$F_n/F_{n-1} = \mathcal{V}_\infty(n) \rtimes_{\Sigma_n} X^{\wedge n} \longrightarrow \tilde{F}_n/\tilde{F}_{n-1} = \mathcal{V}_\infty \rtimes \mathcal{C}_\infty(n) \rtimes_{\Sigma_n} X^{\wedge n}$$

Because of the **homology stability assumption**, this is an  $H_*$ -isomorphism. Hence, by the Whitehead theorem,

$$\mathcal{G}(\tau) \times \mathcal{G}(\pi): \mathcal{G}(\mathbb{V}(X)) \xrightarrow{\simeq} \mathcal{G}(\mathbb{V}(*)) \times \mathcal{G}(\mathbb{C}_\infty(X))$$

Recall:  $\mathcal{G}(\mathbb{C}_\infty(X)) \simeq \Omega^\infty \Sigma^\infty(X)$

### Step 3.: Functorial construction of $\Omega^\infty$ -space

**Bar construction** for monad  $\mathbb{V}$ ,  $\mathbb{V}$ -algebra  $X$ , and  $\mathbb{V}$ -functor  $F$

$$B_\bullet(F, \mathbb{V}, X) := \{q \mapsto F(\mathbb{V}^q(X))\}$$

1.  $|B_\bullet(GF, \mathbb{V}, X)| \simeq |GB_\bullet(F, \mathbb{V}, X)|$  for any functor  $G$
2.  $|B_\bullet(\mathbb{V}, \mathbb{V}, X)| \simeq X$
3.  $|B_\bullet(F, \mathbb{V}, \mathbb{V}(X))| \simeq F(X)$
4. If  $\delta: \mathbb{V} \rightarrow \mathbb{W}$  is a natural transformation of monads, then  $\mathbb{W}$  is a  $\mathbb{V}$  functor and  $B_\bullet(\mathbb{W}, \mathbb{V}, X)$  is a simplicial  $\mathbb{W}$ -algebra.

**Claim:** The assignment  $X \mapsto |GB_\bullet(\mathbb{C}_\infty, \mathbb{V}, X)|$  defines a functor from  $\mathcal{V}$ -spaces to  $\Omega^\infty$ -spaces.

Proof:

$$\begin{array}{ccc}
 |\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{V}, X)| & \longrightarrow & |\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{V}, Y)| \\
 \downarrow & & \downarrow \\
 |\mathcal{G}B_{\bullet}(\Omega^{\infty}\Sigma^{\infty}, \mathbb{V}, X)| & \longrightarrow & |\mathcal{G}B_{\bullet}(\Omega^{\infty}\Sigma^{\infty}, \mathbb{V}, Y)| \\
 \uparrow & & \uparrow \\
 |\mathcal{G}\Omega^{\infty}B_{\bullet}(\Sigma^{\infty}, \mathbb{V}, X)| & \longrightarrow & |\mathcal{G}\Omega^{\infty}B_{\bullet}(\Sigma^{\infty}, \mathbb{V}, Y)| \\
 \uparrow & & \uparrow \\
 |\Omega^{\infty}B_{\bullet}(\Sigma^{\infty}, \mathbb{V}, X)| & \longrightarrow & |B_{\bullet}(\Omega^{\infty}\Sigma^{\infty}, \mathbb{V}, Y)| \\
 \downarrow & & \downarrow \\
 \Omega^{\infty}|B_{\bullet}(\Sigma^{\infty}, \mathbb{V}, X)| & \longrightarrow & \Omega^{\infty}|B_{\bullet}(\Sigma^{\infty}, \mathbb{V}, Y)|
 \end{array}$$

We used the recognition principle for  $\mathbb{C}_{\infty}$ -algebras.

#### Step 4.: Restoring $\mathcal{G}(\mathbb{V}(*))$

**Claim:** For any  $\mathcal{V}$ -space  $X$ , there is a homotopy fibration sequence

$$\mathcal{G}\mathbb{V}(\ast) \rightarrow |\mathcal{G}B_{\bullet}(\mathbb{V}, \mathbb{V}, X)| \rightarrow |\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{V}, X)|.$$

**Claim:** For any  $\mathcal{V}$ -space  $X$ , there are weak homotopy equivalences

$$|\mathcal{G}B_{\bullet}(\mathbb{V}, \mathbb{V}, X)| \xleftarrow{\simeq} \mathcal{G}X \xrightarrow{\simeq} |\mathcal{G}B_{\bullet}(\mathbb{C}_{\infty}, \mathbb{V}, \mathbb{V}(\ast) \times X)|$$

where  $\mathbb{V}(\ast) \times X$  has the diagonal  $\mathcal{V}$ -space structure.