

The blow up section family

Brustenga i Moncusí, Laura brust@mat.uab.cat

Universitat Autònoma de Barcelona

Introduction

A *cluster of r points* of a scheme X is a sequence of points (p_1, \dots, p_r)

$$\begin{array}{ccccccc} X & \longleftarrow & \text{bl}_{p_1} X & \longleftarrow & \dots & & \\ \Psi & & \Psi & & \dots & & \\ p_1 & & p_2 & & \dots & & \end{array}$$

A *cluster of r sections* of a family $X \rightarrow S$ is a sequence of sections $(\sigma_1, \dots, \sigma_r)$

$$\begin{array}{ccccccc} & & \sigma_1 & & \sigma_2 & & \dots \\ & \searrow & & \searrow & & \searrow & \\ S & \longleftarrow & X & \longleftarrow & \text{bl}_{\sigma_1} X & \longleftarrow & \dots \end{array}$$

A *family of clusters of r points* parametrised by a scheme T

$$\begin{array}{ccccccc} & & t_1 & & t_2 & & \dots \\ & \searrow & & \searrow & & \searrow & \\ T & \longleftarrow & T \times X & \longleftarrow & \text{bl}_{t_1}(T \times X) & \longleftarrow & \dots \end{array}$$

A *family of clusters of r sections* parametrised by a scheme T

$$\begin{array}{ccccccc} & & t_1 & & t_2 & & \dots \\ & \searrow & & \searrow & & \searrow & \\ T \times S & \longleftarrow & T \times X & \longleftarrow & \text{bl}_{t_1}(T \times X) & \longleftarrow & \dots \end{array}$$

When $X \rightarrow S$ smooth, families of r -clusters form a functor

$$\mathcal{C}_r: \text{Sch}_{\mathbb{k}} \rightarrow \text{Set}.$$

Theorem 1 (Existence)

If X is quasiprojective and S is proper, then \mathcal{C}_r is representable by a scheme Cl_r .

Theorem 2 (Structure)

If X is quasiprojective and S is proper and smooth, then there is a morphism

$$F: \text{Cl}_{r+1} \rightarrow \text{Cl}_r \times_{\text{Cl}_{r-1}} \text{Cl}_r$$

and a stratification by locally closed subschemes

$$\text{Cl}_r \times_{\text{Cl}_{r-1}} \text{Cl}_r = \sqcup_P D_P = \Delta_{\text{Cl}_r} \sqcup (\sqcup_P D_P)$$

such that, for every irreducible component $C \subseteq \text{Cl}_{r+1}$,

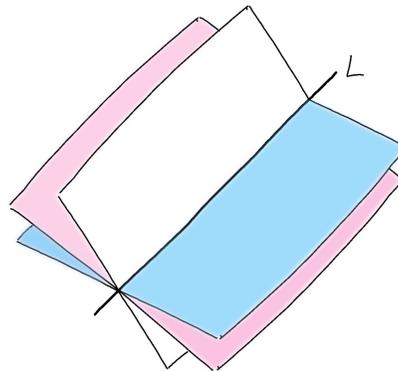
- $F(C) \subseteq \overline{D_P}$ for some (obviously unique) D_P ,
- if $D_P \neq \Delta_{\text{Cl}_r}$, then $F|_C: C \rightarrow \overline{D_P}$ is a blow up along an ideal which fails to be Cartier only along the diagonal $\Delta_{\text{Cl}_r} \cap \overline{D_P}$.

An example

The family (a pencil of planes):

$$\begin{array}{ccc} \pi: \mathbb{P}_{\mathbb{k}}^3 \setminus L & \longrightarrow & \mathbb{P}_{\mathbb{k}}^1 \\ (x:y:z:w) & \longrightarrow & (x:y) \end{array}$$

where L is the line $V(x, y)$. Set $X = \mathbb{P}_{\mathbb{k}}^3 \setminus L$.



Cl_1 parametrises rational curves of degree one, that is lines $R \subseteq X$.

$$\text{Cl}_1 = \mathbb{G}(1, 3) \setminus \{\text{lines intersecting } L\} \cong \mathbb{A}^4$$

Cl_2 parametrises couples of sections (σ, τ) where

- $\sigma: \mathbb{P}_{\mathbb{k}}^1 \rightarrow X$ parametrises a line $R \subseteq X$.
- $\tau: \mathbb{P}_{\mathbb{k}}^1 \rightarrow \text{bl}_{\sigma} X$ is a section of $\text{bl}_{\sigma} X \rightarrow X \rightarrow \mathbb{P}_{\mathbb{k}}^1$

Notation:

$Z \subseteq \text{Cl}_1 \times \text{Cl}_1$ cl. subscheme parametrising lines that intersect.

$E_{\sigma} \subseteq \text{bl}_{\sigma} X$ the exceptional divisor.

- Observation: when τ *does* factorises through E_{σ} , it is a section of

$$\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1 \cong E_{\sigma} \rightarrow \text{bl}_{\sigma} X \rightarrow X \rightarrow \mathbb{P}_{\mathbb{k}}^1.$$

It corresponds to a curve

$$C_n \subseteq \mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1 \text{ of bidegree } (1, n).$$

Such curves are parametrised by open subschemes

$$V_n \rightarrow \mathbb{P}^1 \times \mathbb{P}^n$$

- Observation: for $n = 0, 1$, the curves C_n are respectively flat limit of two intersecting lines or two non-intersecting lines.

$$\text{Cl}_2 = W \sqcup U \sqcup (\sqcup_{n \geq 2} V_n)$$

with

$$W = (Z \setminus \Delta_{\text{Cl}_1}) \cup V_0 \rightarrow Z \quad \text{and}$$

$$U = ((\text{Cl}_1 \times \text{Cl}_1) \setminus Z) \cup V_1 \rightarrow \text{Cl}_1 \times \text{Cl}_1.$$

The blow up section family

Consider schemes X, S and a closed subscheme Z of $X \times S$. The *blow up section family of the projection $X \times S \rightarrow S$ along Z* is a morphism $b: \mathfrak{X} \rightarrow X$ such that $(b \times \text{Id}_S)^{-1}(Z) \subseteq \mathfrak{X} \times S$ is an effective Cartier divisor

$$\begin{array}{ccccc} (b \times \text{Id}_S)^{-1}(Z) & \xrightarrow{\text{Cartier}} & \mathfrak{X} \times S & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow b \times \text{Id}_S & & \downarrow b \\ Z & \xrightarrow{\text{cl.emb.}} & X \times S & \longrightarrow & X \end{array}$$

and b satisfies the following universal property: For every morphism $f: X' \rightarrow X$ such that $(f \times \text{Id}_S)^{-1}(Z) \subseteq X' \times S$ is an effective Cartier divisor, there is a unique morphism $g: X' \rightarrow \mathfrak{X}$ such that $f = g \circ b$.

Theorem 3 (Existence)

If X is quasiprojective and S is projective and integral, then the blow up section family of $X \times S \rightarrow S$ along Z exists.

Theorem 4 (Structure)

If X is quasiprojective and S is projective, integral and smooth, then there is a stratification by locally closed subschemes with a distinguished stratum X' (which is a closed subscheme)

$$X = \sqcup_i X_i = X' \sqcup (\sqcup_j X_j)$$

such that the morphism

$$b|_{\mathfrak{X} \setminus b^{-1}(X')}: \mathfrak{X} \setminus b^{-1}(X') \rightarrow X \setminus X'$$

factorises through $\sqcup_j X_j$ and such a corestriction is an isomorphism with its image.

Examples

- When S is the base field, we retrieve the classic blow up and Theorem 3 just says that a blow up is an isomorphism away of its centre.
- From the example, the morphism

$$b: W \sqcup U \rightarrow \text{Cl}_1 \times \text{Cl}_1$$

is the blow up sections family of $\mathbb{P}_{\mathbb{k}}^1 \times (\text{Cl}_1 \times \text{Cl}_1) \rightarrow \mathbb{P}_{\mathbb{k}}^1$ along a suitable closed subscheme. The distinguished stratum is Δ_{Cl_1} and

$$(W \sqcup U) \setminus b^{-1}(\Delta_{\text{Cl}_1}) = (Z \setminus \Delta_{\text{Cl}_1}) \sqcup ((\text{Cl}_1 \times \text{Cl}_1) \setminus Z)$$