Moduli spaces of unstable curves

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(based on work of Joshua Jackson and others)

Classification problems in geometry

Ingredients:
(1) **objects** (e.g. compact Riemann surfaces/nonsingular complex projective curves);
(2) **equivalence relation** $\sim$ (e.g. biholomorphism/isomorphism);
(3) **families** of objects parametrised by base spaces S (typically $\pi : \mathcal{X} \to S$ with $\pi^{-1}(s)$ the object parametrised by $s \in S$).

Fix any discrete invariants (e.g. genus of compact Riemann surface/nonsingular projective curve). Then

moduli space $= \{\text{objects}\} / \sim$ with nice geometric structure.

E.g. $\mathcal{M}_g = \{$nonsingular curves of genus $g\}$/isomorphism

Moduli spaces often arise as (parameter space)/(group action).

E.g. (1) clock: $S^1 \cong \mathbb{R}/\mathbb{Z}$
where $S^m = \{(y_0, y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_0^2 + y_1^2 + \cdots + y_m^2 = 1\}$
is a sphere of dimension m and $\mathbb{Z}$ acts on $\mathbb{R}$ by translation.

E.g. (2) complex projective space $\mathbb{P}^n$
$$\begin{aligned} &\cong S^{2n+1}/S^1 \\ &\cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* \end{aligned}$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (under multiplication) acts as scalar multiplication on $\mathbb{C}^{n+1} \setminus \{0\}$.

C projective curve (over $\mathbb{C}$) of genus $g \geqslant 2$
C is **stable** iff its singularities are at worst nodes and its automorphism group is finite.

$$\overline{\mathcal{M}}_g = \{\text{stable curves of genus } g\} \,/\, \text{isomorphism}$$

is a projective variety containing $\mathcal{M}_g$ as an open subset.

$\overline{\mathcal{M}}_g = \mathcal{K}^s / SL(r+1)$ is the quotient by $SL(r+1)$ in the sense of geometric invariant theory (GIT) of the closure

$$\mathcal{K} = \overline{\{k-\text{canonically embedded nonsing curves of genus } g\}}$$

in the Hilbert scheme of curves in $\mathbb{P}^r$ with Hilbert polynomial $P(m) = dm + 1 - g$ for $r = (2k-1)(g-1) - 1$ and $d = 2k(g-1)$, where $k >> 1$ and $\mathcal{K}^s$ is the stable subset of $\mathcal{K}$.

A singular curve C has a canonical 'resolution of singularities', its normalisation $p : \tilde{C} \to C$. Here $\tilde{C}$ is a nonsingular curve and p is surjective and an isomorphism away from the singular locus of C.

Rosenlicht–Serre description of singular curves C via the normalisation $p : \tilde{C} \to C$: Let R be the equivalence relation on $Y = \tilde{C}$ defined by $p : \tilde{C} \to C$ with associated homeomorphism $Y/R \to C$ where Y/R has the co-finite topology. For $z \in Y/R$ let

$$\mathcal{O}_z = \bigcap_{p(y)=z} \mathcal{O}_{y,Y} \qquad \text{(semi-local ring)}$$

with radical $\mathfrak{r}_z$. Let $\mathcal{O}'_z$ be $\mathcal{O}_z$ if $z \notin$ (Sing(C) and $\mathbb{C} + \mathfrak{r}_z \supseteq \mathcal{O}'_z \supseteq \mathbb{C} + \mathfrak{r}_z^n$ for some $n >> 1$. Then the ringed space $(Y/R, \mathcal{O}')$ is always a projective curve, and is isomorphic to the original curve C for appropriate choices of $n >> 1$ and of $\mathcal{O}'_z$ when $z \in$ (Sing(C).

Unibranch case ($p : \tilde{C} \to C$ bijective):

Sherwood Ebey (1964) used the Rosenlicht–Serre construction to classify unibranched singularities using slices, given by constructible subsets, for a non-reductive linear algebraic group action on a variety (and showed in general $\exists$ nontrivial moduli). The same groups were used by Demailly in the 1990s to study jet differentials:

X *complex manifold*

$J_k \to X$ *bundle of k-jets of holomorphic curves* $f : (\mathbb{C}, 0) \to X$

Under composition modulo t^{k+1} we have a group $\mathbb{G}_k$ acting on J_k whose elements are k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$

$$t \mapsto \phi(t) = a_1 t + a_2 t^2 + \ldots + a_k t^k, \quad a_j \in \mathbb{C}, a_1 \neq 0.$$

This reparametrisation group $\mathbb{G}_k$ is isomorphic to the group of matrices

$$\left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ 0 & a_1^2 & \dots & \\ & & \dots & \\ 0 & 0 & \dots & a_1^k \end{pmatrix} : a_1 \in \mathbb{C}^*, a_2, \dots a_k \in \mathbb{C} \right\}.$$

$\mathbb{G}_k$ has a subgroup $\mathbb{C}^*$ (represented by $\phi(t) = a_1 t$) and a unipotent subgroup $\mathbb{U}_k$ (represented by $\phi(t) = t + a_2 t^2 + \dots + a_k t^k$) such that

$$\mathbb{G}_k \cong \mathbb{U}_k \rtimes \mathbb{C}^*;$$

$\mathbb{U}_k$ is its unipotent radical.

Question: Can we modify the construction of

$$\overline{\mathcal{M}}_g = \mathcal{K}^s/SL(r+1)$$

as the GIT quotient by $G = SL(r+1)$ of $\mathcal{K}$ to find discrete invariants of unstable curves such that moduli spaces of curves with these invariants fixed can also be constructed by GIT methods?

Answer: We need to use geometric invariant theory for non-reductive group actions, rather than classical GIT, even though $G = SL(r+1)$ is reductive.

Classical GIT tells us that $\mathcal{K}$ has a stratification

$$\mathcal{K} = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

indexed by a finite subset $\mathcal{B}$ of a +ve Weyl chamber for $SU(r+1)$, with (i) $S_0 = \mathcal{K}^s$, and for each $\beta \in \mathcal{B}$
(ii) the closure of S_β is contained in $\bigcup_{\gamma \geqslant \beta} S_\gamma$,
(iii) $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$ where P_β is a parabolic subgroup of $G = SL(r+1)$ and Y_β^{ss} is an open subset of a projective subscheme $\overline{Y}_\beta$ of $\mathcal{K}$, determined by the action of the Levi subgroup of P_β with respect to a twisted linearisation.

To construct a quotient of (an open subset of) S_β by G we can study the linear action on $\overline{Y}_\beta$ of the parabolic subgroup P_β, twisted appropriately.

Recall: moduli spaces in algebraic geometry are often constructed as quotients of varieties by actions of linear algebraic groups.

Assume for simplicity that we are working over $\mathbb{C}$. A linear algebraic group G is a semi-direct product of a unipotent group, which is its unipotent radical, by a reductive group.

Example: The automorphism group of the weighted projective plane $\mathbb{P}(1,1,2) = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^3$ with weights 1,1 and 2, is

$$\mathrm{Aut}(\mathbb{P}(1,1,2)) \cong R \ltimes U$$

$$\text{with } R \cong GL(2) \times_{\mathbb{C}^*} \mathbb{C}^* \cong GL(2) \text{ reductive}$$

$$U \cong (\mathbb{C}^+)^3 \text{ unipotent}$$

where $(x,y,z) \mapsto (x, y, z + \lambda x^2 + \mu xy + \nu y^2)$ for $(\lambda, \mu, \nu) \in \mathbb{C}^3$.

Mumford's **G***eometric* **I***nvariant* **T***heory* (1960s)

G complex reductive group
X complex projective variety acted on by G

We require a **linearisation** of the action (i.e. an ample line bundle L on X and a lift of the action to L; think of $X \subseteq \mathbb{P}^n$ and the action given by a representation $\rho : G \to GL(n+1)$).

$$\begin{array}{ccccl} X & \rightsquigarrow & A(X) & = & \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}_X \\ | & & & & = \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\ | & & \cup| & & \\ \downarrow & & & & \\ X//G & \Leftarrow & A(X)^G & & \text{algebra of invariants} \end{array}$$

G reductive implies that $A(X)^G$ is a *finitely generated* graded complex algebra so that $X//G = \text{Proj}(A(X)^G)$ is a projective variety.

The rational map $X - - \to X//G$ fits into a diagram

$$\begin{array}{rcccl} & X & - - \to & X//G & \text{cx proj variety} \\ & \bigcup & & \| & \\ \text{semistable} & X^{ss} & \stackrel{\text{onto}}{\longrightarrow} & X//G & \\ & \bigcup & & \bigcup & \text{open} \\ \text{stable} & X^s & \longrightarrow & X^s/G & \end{array}$$

where the morphism $X^{ss} \to X//G$ is G-invariant and surjective.

Topologically $\boxed{X//G = X^{ss}/\sim}$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$.

A partial desingularisation of $X//G$

G complex reductive group
X complex projective variety acted on linearly by G

There is a partial desingularisation $\tilde{X}//G = \tilde{X}^{ss}/G$ of $X//G$ which is a geometric quotient by G of an open subset $\tilde{X}^{ss} = \tilde{X}^{s}$ of a G-equivariant blow-up $\tilde{X}$ of X.

$\tilde{X}^{ss}$ is obtained from X^{ss} by successively blowing up along the subvarieties of semistable points stabilised by reductive subgroups of G of maximal dimension and then removing the unstable points in the resulting blow-up.

X nonsingular $\Rightarrow \tilde{X}//G$ is an orbifold.

Good case: $X^{ss} = X^s \neq \emptyset$

Then

$$X^s/G = X//G = \text{Proj}(A(X)^G)$$

is a projective variety, geometric quotient of X^s.

More generally: $X^s \neq \emptyset$

Then

$$\begin{array}{rcl} & \tilde{X}//G & \text{proj variety} \\ & \| & \\ & \tilde{X}^s/G & \text{geom quotient} \\ \text{open} & \bigcup & \\ & X^s/G & \text{geom quotient} \end{array}$$

So what happens if G is not reductive?
Problem: We can't define a projective variety

$$X//G = \mathrm{Proj}(A(X)^G)$$

because $A(X)^G$ is not necessarily finitely generated.

Question: Can we define a sensible 'quotient' variety $X//G$ when G is not reductive? If so, can we understand it geometrically?

Answer: We can define open subsets X^s ('stable points') and X^{ss} ('semistable points') with a geometric quotient $X^s \to X^s/G$ and an 'enveloping quotient' $X^{ss} \to X//G$. BUT $X//G$ is **not necessarily projective** and $X^{ss} \to X//G$ is **not necessarily onto**.

Defn: Call a unipotent linear alg group U **graded unipotent** if there is a homomorphism $\lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $Lie(U)$ all **strictly positive**. Then let

$$\hat{U} = U \rtimes \mathbb{C}^* = \{(u,t) : u \in U, t \in \mathbb{C}^*\}$$

with multiplication $(u,t) \cdot (u',t') = (u(\lambda(t)(u')), tt')$.

Suppose that $\hat{U}$ acts linearly (with respect to an ample line bundle L) on a projective variety X. We can twist the action of $\hat{U}$ by any (rational) character. If we are willing to twist by an appropriate character then GIT for the $\hat{U}$ action is nearly as well behaved as in the classical case for reductive groups.

Thm: (Berczi, Doran, Hawes, K) *Let U be graded unipotent acting linearly on a projective variety X, and suppose that the action extends to $\hat{U} = U \rtimes \mathbb{C}^*$. Suppose also that*

$$(*) \quad x \in X^{\mathbb{C}^*}_{\min} \Rightarrow \dim \mathrm{Stab}_U(x) = \min_{y \in X} \dim \mathrm{Stab}_U(y)$$

where $X^{\mathbb{C}^*}_{\min}$ is the union of connected components of $X^{\mathbb{C}^*}$ where $\mathbb{C}^*$ acts on the fibres of L with minimum weight. Twist the action of $\hat{U}$ by a (rational) character so that 0 lies in the lowest bounded chamber for the $\mathbb{C}^*$ action on X. Then
(i) the ring $A(X)^{\hat{U}}$ of $\hat{U}$-invariants is **finitely generated**, so that $X//\hat{U} = \mathrm{Proj}(A(X)^{\hat{U}})$ is **projective**;
(ii) $X//\hat{U}$ is a **geometric quotient** of $X^{ss,\hat{U}} = X^{s,\hat{U}}$ by $\hat{U}$.

Moreover, even without condition (*) there is a projective completion of $X^{s,\hat{U}}/\hat{U}$ which is a geometric quotient by $\hat{U}$ of an open subset $\tilde{X}^{ss}$ of a $\hat{U}$-equivariant blow-up $\tilde{X}$ of X.

Recall that when G is reductive and acts linearly on a projective variety X, it has a stratification

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

indexed by a finite subset $\mathcal{B}$ of a +ve Weyl chamber, with
(i) $S_0 = X^{ss}$, and for each $\beta \in \mathcal{B}$
(ii) the closure of S_β is contained in $\bigcup_{\gamma \geqslant \beta} S_\gamma$,
(iii) $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$ where P_β is a parabolic subgroup of G and Y_β^{ss} is an open subset of a projective subvariety $\overline{Y}_\beta$ of X.

$P_\beta = U_\beta \rtimes L_\beta$, where its unipotent radical U_β is graded by a central 1-parameter subgroup $\mathbb{C}^* \to L_\beta$ of its Levi subgroup L_β. To construct a quotient of (an open subset of) S_β by G we can study the linear action on $\overline{Y}_\beta$ of the parabolic subgroup P_β, twisted as in the previous theorem, and quotient first by $\hat{U}_\beta$ and then by the residual action of the reductive group $P_\beta/\hat{U}_\beta = L_\beta/\mathbb{C}^*$.

Moduli spaces of sheaves of fixed Harder–Narasimhan type over a nonsingular projective variety W

There is a well-known construction due to Simpson of the moduli space of semistable pure sheaves on W of fixed Hilbert polynomial as the GIT quotient of a linear action of a special linear group G on a scheme Q (closely related to a quot-scheme) which is G-equivariantly embedded in a projective space. This construction can be chosen so that elements of Q which parametrise sheaves of a fixed Harder–Narasimhan type form a stratum in the stratification of Q associated to the linear action of G, at least modulo taking connected components of strata (Hoskins). One can then try to use non-reductive GIT for the associated linear action of a parabolic subgroup of G, appropriately twisted, to construct moduli spaces of sheaves of fixed Harder–Narasimhan type over W.

Moduli spaces of unstable curves of fixed ‘type’

Similarly we can define the ‘Rosenlicht–Serre type’ of a projective curve, measuring to some extent how unstable it is.

Non-reductive GIT for linear actions of suitable non-reductive linear algebraic groups with internally graded unipotent radicals can then be used to construct moduli spaces of unstable curves of fixed Rosenlicht–Serre type.
(Joshua Jackson, Oxford thesis, 2018)