Exceptional Collections of Toric Varieties Associated to Root Systems

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Exceptional Collections & Semiorthogonal Decompositions

Let $D^b(X)$ denote the *bounded derived category of (quasi-)coherent sheaves* of a variety X over an arbitrary field k. (In particular, k is not required to be algebraically closed.)

Definition. *[1] An object E of a k-linear derived category $D^b(X)$ is called* exceptional *if*

$$\mathrm{Ext}^{\ell}_{D^b(X)}(E,E) = \mathrm{Hom}(E, E[\ell]) = \begin{cases} \text{a division } k\text{-algebra} & \text{if } \ell = 0 \\ 0 & \text{if } \ell \neq 0 \end{cases}$$

An exceptional sequence is a sequence $E_1, \ldots, E_n$ of exceptional objects such that $\mathrm{Ext}^{\ell}_{D^b(X)}(E_i, E_j) = 0$ for all $i > j$ and all ℓ. An exceptional sequence is called full *if $D^b(X)$ is generated by $\{E_i\}$, in other words, any full triangulated subcategory containing all objects E_i is equivalent to $D^b(X)$.*

Note that this varies agrees with the usual definition: when k is algebraically closed we need $\mathrm{Hom}(E, E[\ell]) = k$ when $\ell = 0$.

Definition. *A sequence of full admissible triangulated subcategories*

$$\mathcal{D}_1, \ldots, \mathcal{D}_n \subset D^b(X)$$

is semi-orthogonal *if for all $i > j$*

$$\mathcal{D}_j \subset \mathcal{D}_i^{\perp}.$$

Such a sequence defines a semi-orthogonal decomposition *of $D^b(X)$ if $D^b(X)$ is generated by the $\mathcal{D}_i$.*

Theorem. *[3] Say X has an exceptional collection $\{E_1, E_2, \ldots, E_n\}$. If $\mathcal{E}_i$ is the category generated by each E_i, there is a semi-orthogonal decomposition $D^b(X) = \langle \mathcal{E}_1, \ldots, \mathcal{E}_n, \mathcal{A}\rangle$, where $\mathcal{A}$ is the full subcategory of $D^b(X)$ with objects A such that $\mathrm{Hom}(A, E_i) = 0$ for all i.*

Question. Given a smooth projective toric variety X over an arbitrary field, does X admit a full exceptional collection?

Example. $\mathbb{P}^n_{\mathbb{C}}$ has the following full exceptional collection:

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\rangle.$$

Changing the Base Field

Question. Given a variety defined over the field k with exceptional collection E, what happens to E under base change?

Example. Consider $X := Z(x^2 + y^2 + z^2) \subset \mathbb{P}^2_{\mathbb{R}}$. Notice that $X_{\mathbb{C}} \cong X \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{P}^1_{\mathbb{C}}$, but this is clearly not the case over $\mathbb{R}$ as X has no $\mathbb{R}$-points. X has an exceptional collection $D^b(X) = \langle \mathcal{O}, \mathcal{F}\rangle$ with $\mathrm{End}_{D^b(X)}(\mathcal{F}) \cong \mathbb{H}$. Over $\mathbb{C}$, we see that $\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathcal{O}(1)^{\oplus 2}$.

Definition. *Let X a k-variety, L a finite Galois extension of k, and G the Galois group of the extension. Any $g \in \mathrm{Gal}\, L/k$ defines a morphism of k-schemes $g : X_L \to X_L$ which then gives a functor $g^* : D^b(X_L) \to D^b(X_L)$. An exceptional collection $\mathrm{E} := \{E_1, \ldots, E_n\}$ is* Galois-stable *if we have that $g^* E_i \cong E$ for some $E \in \mathrm{E}$ for all $g \in G$ and $1 \leq i \leq n$.*

Theorem. *[1] (Galois descent for stable collections) Let X and L/k as above, and let $\pi : X_L \to X$ the natural projection map. If X_L admits a full G-stable exceptional collection E of objects of $D^b(X_L)$, then X admits a full exceptional collection of objects F of $D^b(X)$.*

Toric Varieties associated to Root Systems

Given a fixed rank n root system R in an n-dimensional Euclidean space E, let $M(R)$ denote the root lattice of R, and $N(R) := M(R)^{\vee}$. For any set of simple roots Δ of R, we have the following cone:

$$\sigma_{\Delta} := \{v \in N(R)_{\mathbb{Q}} \mid \langle \alpha, v\rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

This is the Weyl chamber associated to our choice of Δ. The Weyl group of R acts freely and transitively on the set of Weyl chambers, so by acting on σ_{Δ} we recover all other Weyl chambers. These will be the maximal cones of our smooth toric variety, $X(R)$.

Example. Consider $R = B_2$. A set of simple roots is $\Delta = \{(1,-1), (0,1)\}$.

Figure 1: Dynkin diagram for B_2 root system

Taking duals, we see that the fundamental Weyl chamber is spanned by $(1,1), (1,0)$. Acting by $\mathrm{Weyl}(B_2) \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2$ gives the following fan:

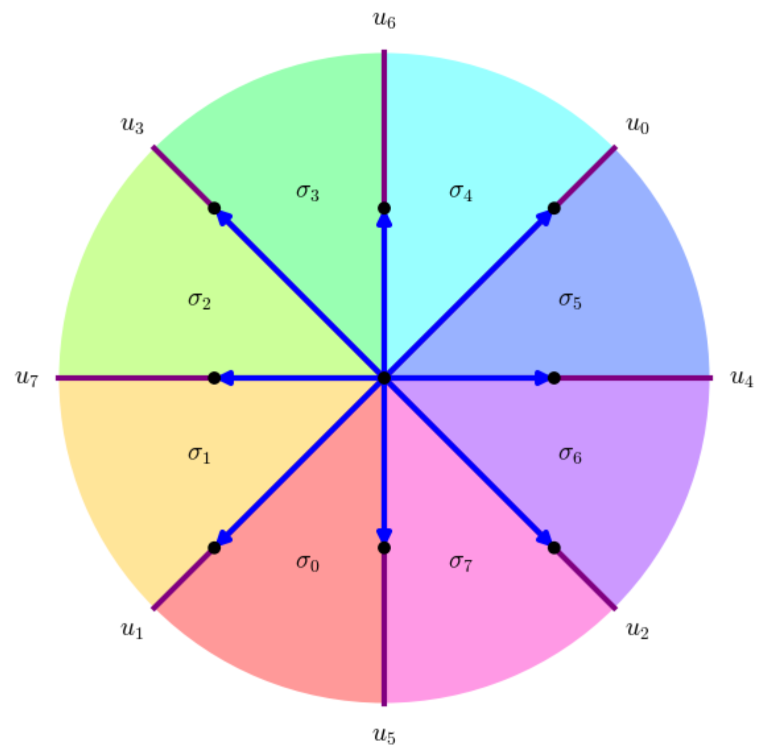

Figure 2: Fan in $N(R)$ for $X(B_2)$.

More examples of the fans of one and two-dimensional toric varieties associated to root systems can be seen below. Notice that $X(A_1) \cong \mathbb{P}^1$, $X(A_2) \cong \mathrm{dP}_6$, and $X(D_2) \cong X(A_1) \times X(A_1) \cong \mathbb{P}^1 \times \mathbb{P}^1$.

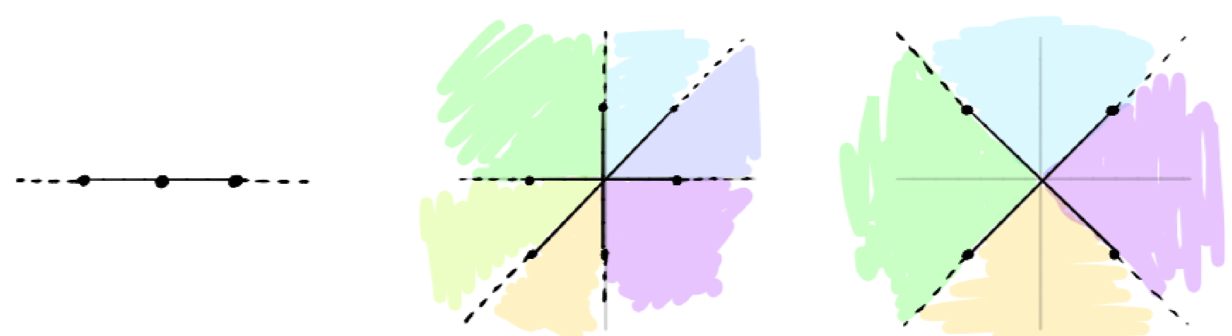

Figure 3: Fans for $X(A_1), X(A_2), and X(D_2)$ respectively.

An Exceptional Collection of $X(B_2)$

Using Orlov's blowup formula, we can obtain an exceptional collection of $X(B_2)$ from that of $X(A_1 \times A_1)$. In this particular case, the exceptional collection that we obtain is invariant under the automorphism group of $\Sigma(B_2)$. This is not the case for general $X(B_n)$.

Exceptional Collections of $X(A_n)$

In 2017, Castravet and Tevelev [4] were able to give an exceptional collection on the Losev-Manin moduli space $\overline{LM}_n$ invariant under the symmetries of the fan of $X(A_n)$. Notice that $\overline{LM}_{n-1} \cong X(A_n)$, [5] and $\mathrm{Weyl}(A_n) \cong S_n$, which makes this a natural result to look towards.

Figure 4: Dynkin Diagram of type A_n.

Theorem. *(Castravet & Tevelev [4])*
$D^b(X(A_n))$ admits the semi-orthogonal decomposition

$$D^b(X(A_n)) = \langle D^b_{cusp}(X(A_n)), \{\pi^* D^b_{cusp}(X(A_k))\}_k, \mathcal{O}\rangle$$

where $k \in \{n-1, ..., 1\}$ ordered by decreasing size.

Definition. *Given a collection of morphisms of smooth projective varieties $\pi_i : X \to X_i$ for $i \in I$, we call an object $E \in D^b(X)$* **cuspidal** *if*

$$R\pi_{i*}E = 0 \quad \text{for every } i \in I.$$

The **cuspidal block** *is the full triangulated subcategory of cuspidal objects $D^b_{cusp}(X) \subset D^b(X)$.*

Exceptional Collections of $X(D_4)$

One might ask whether or not $X(D_n)$ is isomorphic to a nice moduli space as in the previous case of $X(A_n)$. This was shown to be false, [2] as the morphism $X(D_{n-1}) \to X(D_n)$ induced by morphisms of root systems is not flat.

Proposition. *(Ballard, L.) There exist line bundles G_1, G_2, G_3, G_4 on $X(D_4)$ that form an exceptional collection of $D^b_{cusp}(X(D_4))$ that is invariant under the symmetries of the fan $\Sigma(D_4)$. The G_i are pulled back from line bundles on the wonderful compactification associated to a fundamental set of weights.*

In Progress: Construct a full exceptional collection of $D^b(X(D_n))$ that is stable under the actions of $\mathrm{Weyl}(D_n)$ and $\mathrm{Aut}(\Sigma(D_n))$. Then, use Galois-descent to show that $D^b(X(D_n))$ has a full exceptional collection when $X(D_n)$ is considered over an arbitrary base field.

References

[1] Matthew R. Ballard, Alexander Duncan, and Patrick K. McFaddin. On derived categories of arithmetic toric varieties. *arXiv e-prints*, page arXiv:1709.03574, Sep 2017. To appear in Ann. of K-Theory.

[2] Victor Batyrev and Mark Blume. On generalisations of Losev-Manin moduli spaces for classical root systems. *arXiv e-prints*, page arXiv:0912.2898, Dec 2009.

[3] A. I. Bondal. Representation of Associative Algebras and Coherent Sheaves. *Izvestiya: Mathematics*, 34:23–42, Feb 1990.

[4] Ana-Maria Castravet and Jenia Tevelev. Derived category of moduli of pointed curves - I. *arXiv e-prints*, page arXiv:1708.06340, Aug 2017.

[5] A. Losev and Y. Manin. New moduli spaces of pointed curves and pencils of flat connections. *Michigan Math J.*, 48:443–472, 2000.

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