Left Invariant Lorentzian Einstein metrics on nilpotent Lie groups

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Introduction

The study of Riemannian left-invariant metrics on Lie groups has played a central role in differential geometry, not only it has given hints on some important results concerning the curvature of Riemannian homogeneous spaces but it also turned out to be very useful for constructing counter-examples to many conjectured results. In his classical work [1], Milnor has given the keypoints for this study and has reviewed and shown several important theorems. Many of the results that he cited in his article followed the same line of reasoning, which is to describe how restrictions on the metric influence the structure of the Lie group and conversely. These are some classical theorems to get a taste on how results are usually stated :

Theorem (Azencott & Wilson [1])

If a connected Lie group G has a left invariant Riemannian metric with all sectional curvatures $K \leq 0$, then it is solvable. If G is unimodular then any such a metric with $K \leq 0$ must actually be flat ($K = 0$).

Theorem [1]

A Lie group with a Riemannian left invariant metric is flat if and only if its Lie algebra splits orthogonally as a direct sum $\mathfrak{b} \oplus \mathfrak{u}$ where $\mathfrak{b}$ is an abelian subalgebra, $\mathfrak{u}$ an abelian ideal and where the linear transformations $\mathrm{ad}_{\mathfrak{b}}$ are skew-symmetric.

Theorem (Wolf [1])

Suppose that the Lie algebra of G is nilpotent, non abelian. Then for any Riemannian left invariant metric on G there exists a direction of strictly negative Ricci curvature and a direction of strictly positive Ricci curvature. In particular, G cannot admit a Riemannian, left invariant, Einstein metric.

Lorentzian geometry and more generally pseudo-riemannian geometry is also a vast area with many of its problems arising from physics, one such example is the problem of classifying 4-dimensional manifolds that admit Lorentzian Einstein metrics which is of fundamental importance in solving space-time equations of general relativity. This had people interested in studying Lie groups with left-invariant pseudo-riemannian metrics as a first step toward getting global statements. In contrast to the classical case, only few results are known in this general setting and even in the Lorentzian case, being the most addressed due to its closeness to the Riemannian case, the results differ drastically from what has already been known and most theorems fail to generalize in a naive way or become false altogether. The main goal of my thesis is to contribute to the area of Lorentzian geometry on Lie groups, the subject of my research is entitled "Left Invariant Lorentzian Einstein metrics on nilpotent Lie groups" in which we seek to give a list of all simply connected nilpotent Lie groups admitting left invariant Lorentzian Einstein metrics. The principal text we rely upon to achieve this task is the article [2] by M.Boucetta in which he resolved the problem for 2-step nilpotent, simply connected Lie groups, so our main work is to generalize the results in this article and to follow similar steps in order to obtain the desired classification.

Generalities on left-invariant pseudo-riemannian metrics

Definition

Let M be smooth manifold. A pseudo-riemannian metric is a smooth symmetric 2-tensor field on M such that the induced symmetric bilinear map g_p on T_pM is non-degenerate for every $p \in M$.

Let G be a Lie group, the left multiplication ℓ_g, $g \in G$, allows to define the notion of left invariant vector fields, which is any smooth vector field X satisfying :

$$T_x\ell_g(X_x) = X_{gx}, \quad x, g \in G$$

The set $\chi(G)^\ell$ of all left invariant vector field on G a Lie subalgebra of $\chi(G)$ for the usual Lie bracket, furthermore the map $\chi(G)^\ell \longrightarrow T_eG$, $X \mapsto X_e$ is an isomorphism. We can thus give T_eG a Lie algebra structure by defining $[X_e, Y_e] := [X, Y]_e$, the couple $(T_eG, [\ ,\ ])$ is called the Lie algebra of G.

Definition

A pseudo-riemannian Lie group is any Lie group G together with a left invariant pseudo-riemannian metric $\langle\ ,\ \rangle$, i.e $\ell_g^*\langle\ ,\ \rangle = \langle\ ,\ \rangle$ for any $g \in G$.

Any left invariant pseudo-riemannian $\langle\ ,\ \rangle$ metric on a Lie group G induce a pseudo-euclidian product $\langle\ ,\ \rangle_e$ on the Lie algebra $\mathfrak{g}$ of G and it has been shown that this is a bijective correspondance. This allows to shift the study of pseudo-riemannian Lie groups to the study of pseudo-euclidian Lie algebras $(\mathfrak{g}, \langle\ ,\ \rangle)$.

Definition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-euclidian Lie algebra. The Levi-Civita product of $\langle\ ,\ \rangle$ is the bilinear operator $L : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by the formula : for $u, v, w \in \mathfrak{g}$,

$$2\langle L_u v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle.$$

It is straightforward to check that $[u, v] = L_u v - L_v u$. For any $u \in \mathfrak{g}$ define $R_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ by $R_u v = L_v u$ then it follows that $\mathrm{ad}_u = R_u - L_u$ (where $\mathrm{ad}_u(v) := [u, v]$).

Definition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$, the curvature operator of $\langle\ ,\ \rangle$ is the linear operator $K : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ given by the formula :

$$K(u, v)w = L_{[u,v]}w - L_u L_v w + L_v L_u w, \quad u, v, w \in \mathfrak{g}.$$

The Ricci curvature of $\langle\ ,\ \rangle$ is the symmetric bilinear form $\mathrm{ric} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ given by the formula :

$$\mathrm{ric}(u, v) = \mathrm{tr}(w \mapsto K(u, w)v), \quad u, v \in \mathfrak{g}.$$

The Ricci operator is the linear operator $\mathrm{Ric} : \mathfrak{g} \longrightarrow \mathfrak{g}$ given by the formula :

$$\langle \mathrm{Ric}(u), v\rangle := \mathrm{ric}(u, v).$$

The scalar curvature of $\langle\ ,\ \rangle$ is the quantity given by $\mathfrak{s} = \mathrm{tr}(\mathrm{Ric})$.

Definition

A pseudo-euclidian Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ is said to be an Einstein Lie algebra if it has an Einstein pseudo-euclidian product, i.e $\mathrm{Ric} = \lambda \mathrm{Id}_{\mathfrak{g}}$ for some $\lambda \in \mathbb{R}$. In the particular case where $\lambda = 0$ we say that $\mathfrak{g}$ is Ricci-flat.

For convenience, put $J_u v := \mathrm{ad}_v^*(u)$ for any $u, v \in \mathfrak{g}$, and write $\mathrm{tr}(\mathrm{ad}_u) := \langle H, u\rangle$. Then :

$$\mathrm{ric}(u, v) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v) - \frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) + \frac{1}{4}\mathrm{tr}(J_u \circ J_v^*) - \frac{1}{2}\langle(\mathrm{ad}_H + \mathrm{ad}_H^*)(u), v\rangle.$$

Now suppose that $\mathfrak{g}$ is a nilpotent Lie algebra, then $\mathfrak{g}$ is unimodular ($H = 0$) and has a vanishing Killing form (for any $u, v \in \mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v) = 0$), in which case :

$$\mathrm{ric}(u, v) = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) + \frac{1}{4}\mathrm{tr}(J_u \circ J_v^*).$$

We define the linear operators $\mathcal{J}_1, \mathcal{J}_2 : \mathfrak{g} \longrightarrow \mathfrak{g}$ by the formulas :

$$\langle \mathcal{J}_1(u), v\rangle = \mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v^*) \quad \text{and} \quad \langle \mathcal{J}_2(u), v\rangle = \mathrm{tr}(J_u \circ J_v^*).$$

So in order to find all nilpotent Lie algebra structures allowing Einstein pseudo-euclidian product we need to solve $-\frac{1}{2}\mathcal{J}_1 + \frac{1}{4}\mathcal{J}_2 = \lambda \mathrm{Id}_{\mathfrak{g}}$. We start with the case of 2-step nilpotent Lie algebras.

2-step nilpotent Lie groups with Lorentzian Einstein metric

We give principal statements of results in [2] :

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Euclidian, 2-step nilpotent Lie algebra. Suppose $\mathrm{ric} = \lambda\langle\ ,\ \rangle$ for some $\lambda \in \mathbb{R}$ then necessarily $\lambda = 0$.

Proposition

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a Lorentzian, 2-step nilpotent Lie algebra. If $\mathfrak{g}$ is Ricci-flat then its center is degenerate.

Theorem

Let $\mathfrak{g}$ be an irreducible Lorentzian 2-step nilpotent Lie algebra. Then $\mathfrak{g}$ is Ricci-flat if and only if $\mathfrak{g}$ is isomorphic to the Lorentzian vector space $\mathbb{R}^{(1,1)} \times \mathbb{R}^p \times \mathbb{R}^{2r} \times \mathbb{R}^q$ and if,

$$(e, \bar{e}), \ (f_1, \ldots, f_p), \ (g_1, \ldots, g_{2r}), \ (h_1, \ldots, h_q)$$

are the canonical basis, respectively, of $\mathbb{R}^{(1,1)}$, $\mathbb{R}^p$, $\mathbb{R}^{2r}$ and $\mathbb{R}^q$, then the Lie brackets are :

$$[\bar{e}, g_i] = a_i e + \sum_{l=1}^{p} x_i^l f_l, \quad i = 1, \ldots, 2r$$

$$[\bar{e}, h_i] = b_i e + \sum_{i=1}^{l} y_i^l f_l, \quad i = 1, \ldots, q$$

$$[g_{2i-1}, g_{2i}] = \lambda_i e, \qquad i = 1, \ldots, r,$$

and the structure coefficients satisfy the following conditions :

- $$\mathrm{span}\{(b_1, \ldots, b_q), (y_1^1, \ldots, y_q^1), \ldots, (y_1^p, \ldots, y_q^p)\} = \mathbb{R}^q.$$
- $0 < \lambda_1 \leq \cdots \leq \lambda_r$ and
$$\sum_{i,l}(x_i^l)^2 + \sum_{i,l}(y_i^l)^2 = \sum_{i=1}^{r}\lambda_i^2.$$

References

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WOMEN IN HOMOTOPY THEORY, 20 March - 22 March 2019, Berlin, Germany