Geometric and combinatorial models for generalised homotopy associativity

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University of Sheffield

Women in Homotopy Theory and Algebraic Geometry II
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Plan

The new work is mostly from the PhD thesis of Gemma Halliwell.

structure	up to homotopy version of...	geometric models	combinatorial models	counted by...

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derived A_∞	combination of above with $d(a.b) \simeq da.b + a.db$	K_{ij} built from K_j and $I^{\wedge n}$	planar trees with gaps	Narayana numbers and generalizations

Classical A_∞ case

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And assume this is homotopy associative: for each triple of points a, b, c in Y, we have the two points $(ab)c$ and $a(bc)$ in Y and a path between them. The (naive) homotopy associativity of the multiplication is given by a map

$$m_3 : Y^3 \times I \to Y.$$

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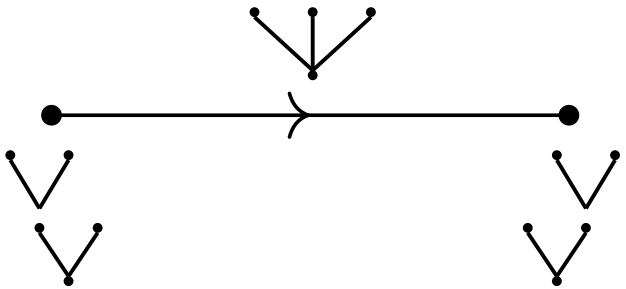
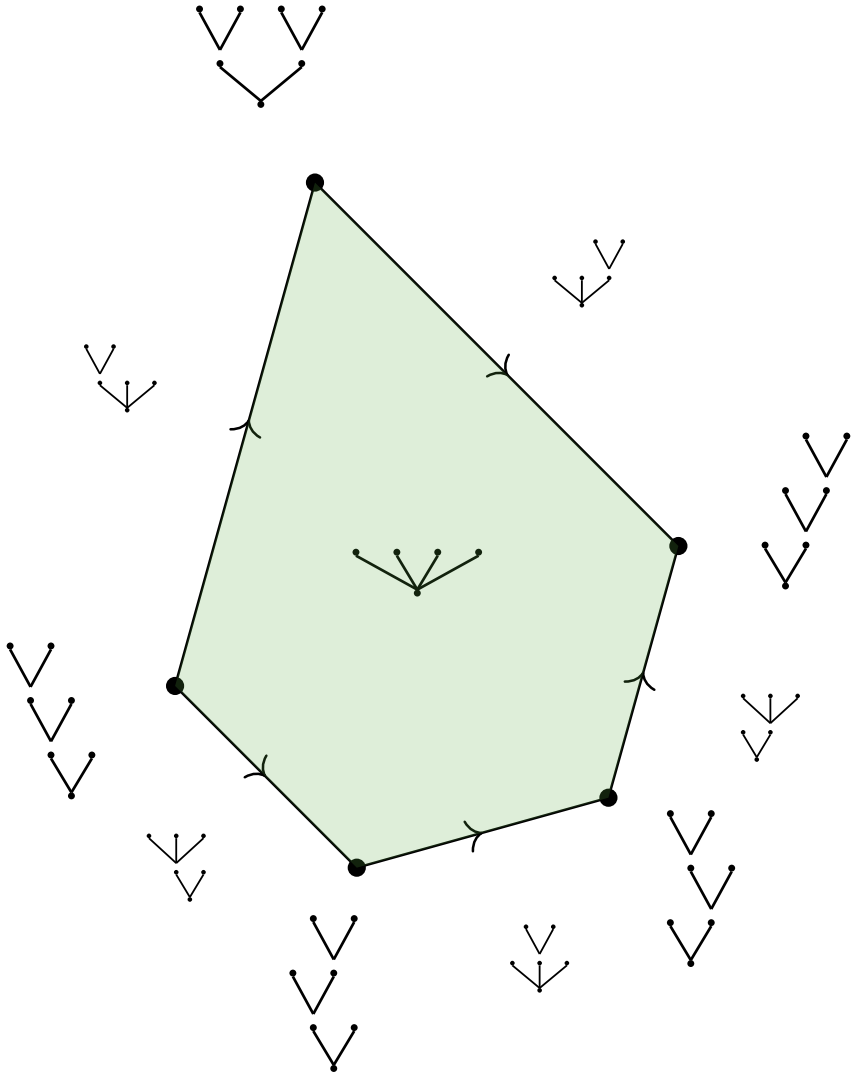
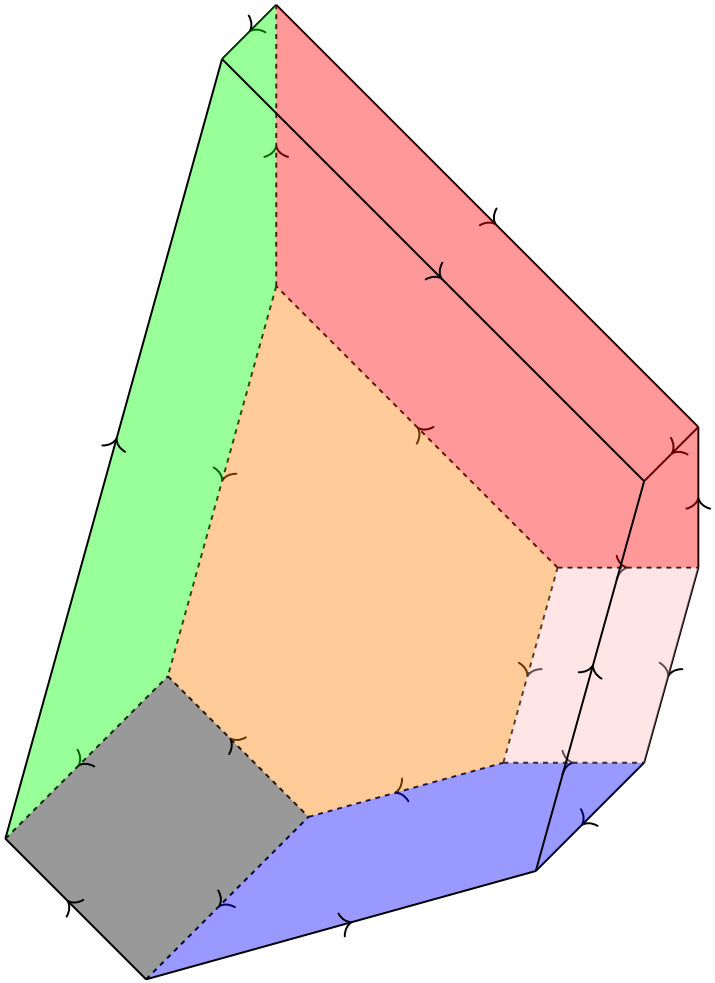
And the process continues, with higher associahedra encoding higher coherences of the associativity.

Associahedra K_3 and K_4 with planar tree labels





Associahedron K_5



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- The $\{K_j\}$ form a (non-symmetric) operad in finite cell complexes, an A_∞-space is an algebra over this operad and $C_*(A_\infty\text{-space})$ is an A_∞-algebra.

Multicomplex case

Definition

A *multicomplex*, C, is an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, with maps $d_i : C \to C$ of bidegree $(i, i-1)$ for $i \geqslant 0$, satisfying

$$\sum_{i+p=u} (-1)^i d_i d_p = 0$$

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Other names for this structure are *twisted complex* and D_∞-algebra (because it is the homotopy coherent version of a differential).

Modelling multicomplexes

We need to move to *based* spaces. Consider a based space Y with a continuous based map $d_1 : Y \to Y$ such that

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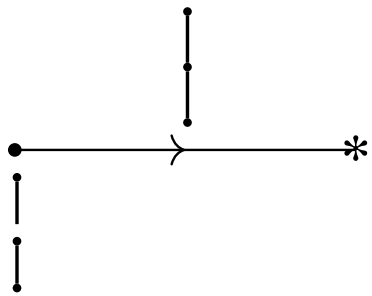
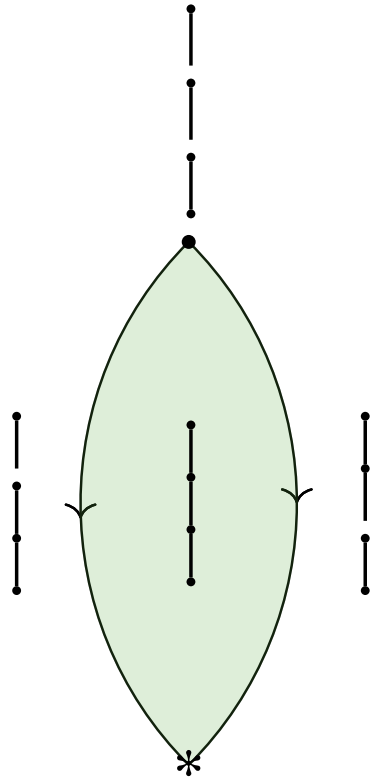
Then d_1^3 is null-homotopic in *two ways*, and there's a higher coherence condition relating these:

$$d_1 d_2 - d_2 d_1 \simeq_{d_3} *.$$

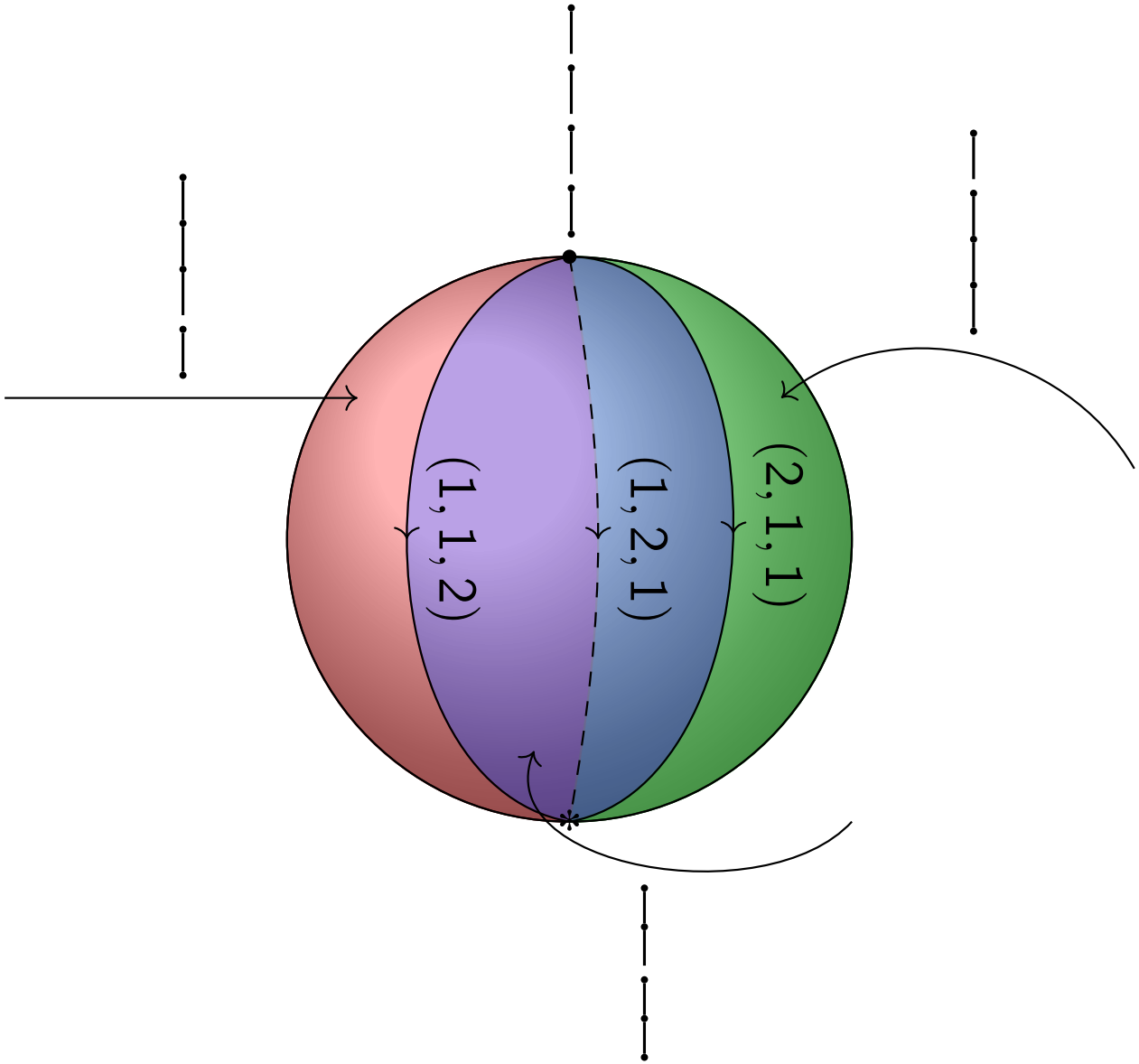
Modelling multicomplexes

So we have $T_2 = I$ for the initial homotopy and then a reduced square T_3 for the next:





The next condition is controlled by T_4, a reduced cube:



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- $C_*(D_\infty\text{-space})$ is a multicomplex.

Derived A_∞ case

Definition

A *derived A_∞-algebra* (or dA_∞-algebra for short) is an $(\mathbb{N}, \mathbb{Z})$-bigraded R-module, A, with R-linear maps

$$m_{ij} : A^{\otimes j} \to A$$

of bidegree $(i, i+j-2)$ for each $i \geqslant 0$, $j \geqslant 1$, satisfying the equations

$$\sum (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0,$$

where the sum is over $u = i + p, v = j + q - 1, j = 1 + r + t$, for all $u \geqslant 0$ and $v \geqslant 1$.

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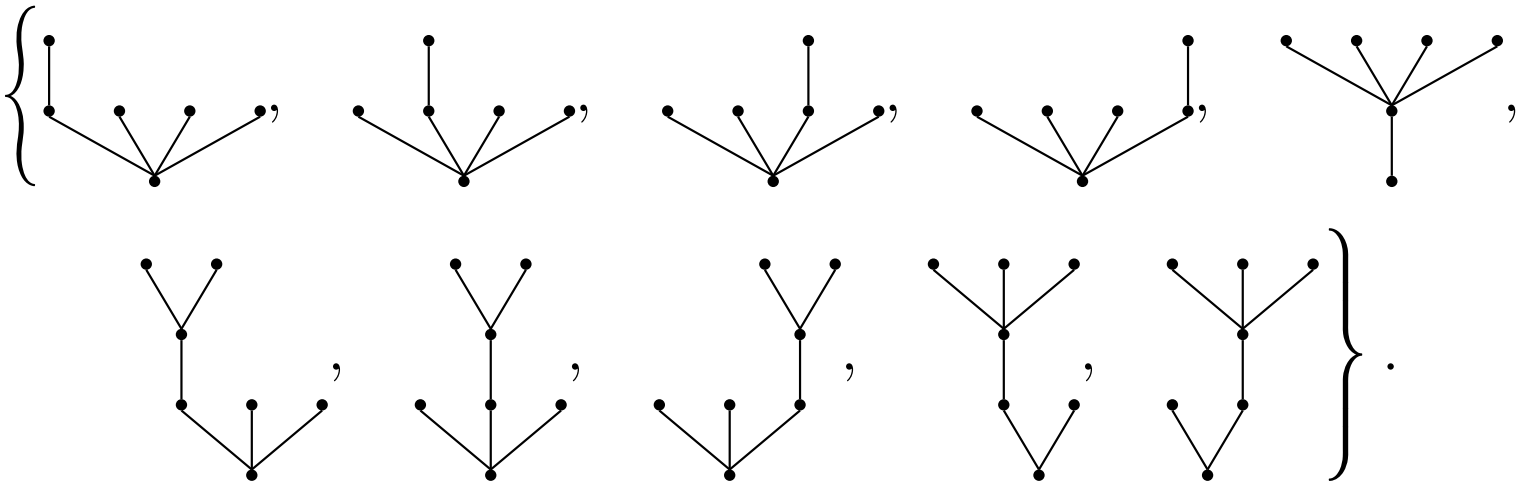
We let K_1 be a point and we define

$$K_{ij} = \bigvee_{t \in \mathcal{T}_{ij}} T_{r+1} \wedge K_{in(v_0)+} \wedge K_{in(v_1)+} \wedge \cdots \wedge K_{in(v_r)+},$$

where t has root vertex v_0 and internal vertices $v_1, \ldots, v_r$.

Example: $\mathcal{T}_{1,4}$ - 4 leaves and 1 node



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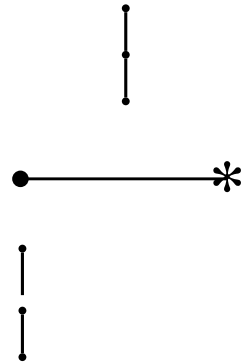
- C_*(algebra over this operad) is a derived A_∞-algebra.

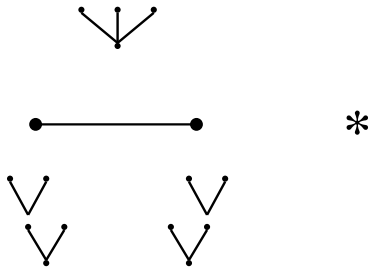
First few K_{ij}s

(a) K_{11}

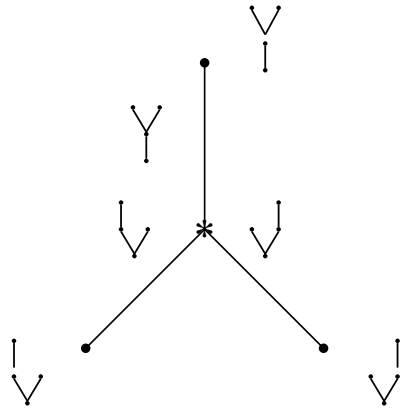
(b) K_{02}



(c) K_{21}

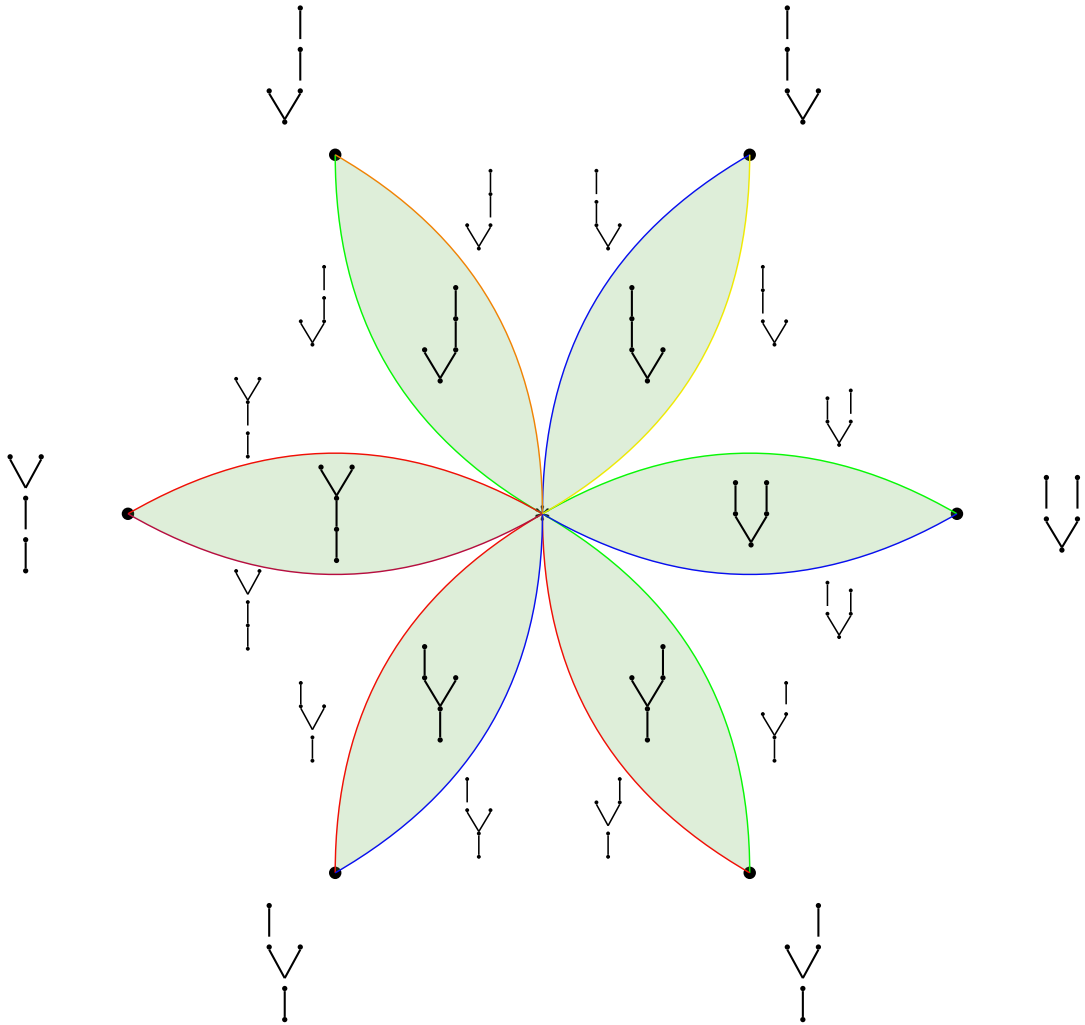


(d) K_{03}

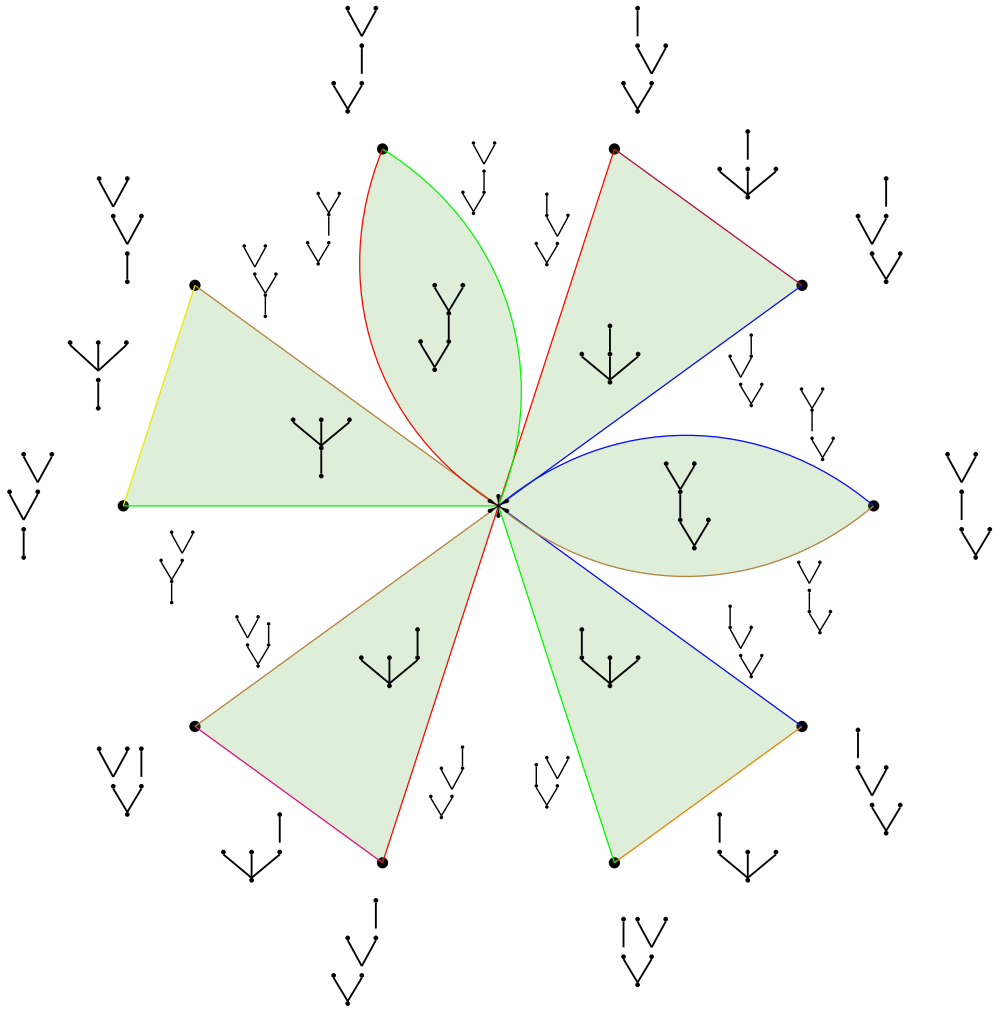


(e) K_{12}

K_{22}



K_{13}



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- Fu-Guan-Otter-Livernet-W: Model structures for special cases of multicomplexes [WiT III project, just getting going]

Thank you for listening!