Goodwillie Calculus II

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Let S denote the ∞-category of spaces. For a small ∞-category C, denote by $[C, S]$ the ∞-category of functors $F\colon C \to S$, that is, presheaves on C^{op}.

Recall that an ∞-*topos* is an accessible left exact localization of $[C, S]$ for some small ∞-category C, where *left exact* means that it preserves finite limits.

If $\mathcal{E}$ is an ∞-topos, then a *modality* on $\mathcal{E}$ is a factorization system $(\mathcal{L}, \mathcal{R})$ such that $\mathcal{L}$ is stable by base change (i.e., pullback along some map).

If F is a left exact localization on an ∞-topos $\mathcal{E}$, then the class $\mathcal{L}$ of F-equivalences and the class $\mathcal{R}$ of F-local maps form a modality. A map $f\colon A \to B$ is called *F-local* if the square $f \to Ff$ is a pullback.

1 Construction of P_n

Let C be a small ∞-category with finite colimits and a terminal object. The main examples are the ∞-categories Fin and Fin_* of *finite spaces* and *finite pointed spaces*. The category Fin is defined as the smallest full subcategory of S which contains the terminal object $*$ and is closed under finite colimits (the closure of $*$ under colimits is S). Then $\mathrm{Fin}_* = [*, \mathrm{Fin}]$.

Recall that a functor $F\colon C \to S$ is *n-excisive* if it carries strongly cocartesian $(n+1)$-cubes in C to cartesian cubes in S. We denote by $[C, S]^{(n)}$ the full subcategory of $[C, S]$ whose objects are the n-excisive functors.

A functor $F\colon C \to S$ is *reduced* if $F(1) = *$. A *spectrum* is a reduced 1-excisive functor $\mathrm{Fin}_* \to S$. If E is a spectrum and we denote $E_n = ES^n$, then E_n is a pointed space such that $E_n \simeq \Omega E_{n+1}$ for all n. More generally, a *spectrum object* in an ∞-category X with finite limits is a reduced 1-excisive functor $\mathrm{Fin}_* \to X$.

If the condition that F be reduced is omitted, then $[\mathrm{Fin}_*, S]^{(1)}$ can be viewed as the ∞-category of *parametrized spectra*. A spectrum parametrized by a space B is a spectrum object in $S_{/B}$.

For a functor $F\colon C \to S$ and $n \geq 0$, define $T_nF\colon C \to S$ as

$$T_nF = \lim_{\emptyset \neq U \in [n]} F(K \star U)$$

with the map $t_n\colon F \to T_nF$ determined by the fact that $K \star \emptyset = K$. Here a finite set U is viewed as an object of C by taking a coproduct of U copies of the terminal object 1, and define $K \star U$ as the colimit of a cone from K to U. For example, if U has two elements, then $K \star U$ is a suspension of K.

Note that T_0F is the constant functor taking the value $F(1)$, and $T_1F = \Omega \circ F \circ \Sigma$.

Then one defines, as in [4],

$$P_nF = \mathrm{colim}(F \to T_nF \to T_nT_nF \to \cdots).$$

As a special case, if F is reduced, then $P_1F = \lim_m \Omega^m \circ F \circ \Sigma^m$.

2 Properties of P_n

The functor P_nF is n-excisive and the canonical map $F \to P_nF$ is universal among maps from F to n-excisive functors. Moreover, the functor $P_n\colon [C,S] \to [C,S]^{(n)}$ is a left exact localization (that is, it commutes with finite limits), and P_n also commutes with colimits.

Therefore, $[C,S]^{(n)}$ is an ∞-topos.

The classes $\mathcal{L}$ of P_n-equivalences and $\mathcal{R}$ of P_n-local maps form a modality, called the *n-excisive modality.*

It is shown in [2, 3] that $[\mathrm{Fin}_*, S]^{(n)}$ classifies *pointed n-nilpotent objects.*

A central result in [1] is a Blakers–Massey-style theorem stating that if K is the pushout of two maps $f\colon F \to G$ and $g \to F \to H$ between functors $C \to S$, and if f is a P_m-equivalence and g is a P_n-equivalence, then the canonical map $F \to G \times_K H$ is a P_{m+n+1}-equivalence. Dually, if F is the pullback of $f\colon H \to K$ and $g\colon G \to K$ and if f is a P_m-equivalence and g is a P_n-equivalence then the canonical map $G \cup_F H \to K$ is a P_{m+n+1}-equivalence.

The proof of this result uses the Yoneda embedding $Y\colon C^{\mathrm{op}} \to [C,S]$ in order to prove that, given a P_i-equivalence f and P_j-equivalence g between functors $C \to S$, the pushout-product $(\Delta f)\square(\Delta g)$ is a P_{i+j+1}-equivalence, where $\Delta f\colon A \to A \times_B A$ if $f\colon A \to B$. The *pushout product* of two maps $u\colon A \to B$ and $v\colon S \to T$ is the canonical map $u\square v\colon (A \times T) \cup_{A\times S} (B \times S) \to B \times T$.

References:

[1] M. Anel, G. Biedermann, E. Finster, A. Joyal, Goodwillie's calculus of functors and higher topos theory, *J. Topology* **11** (2018), 1100–1132.

[2] M. Anel, G. Biedermann, E. Finster, A. Joyal, Left-exact localizations of ∞-topoi I: Higher sheaves, arXiv:2101.02791v4 [math.CT] (2022).

[3] M. Anel, G. Biedermann, E. Finster, A. Joyal, Left-exact localizations of ∞-topoi II: Grothendieck topologies, arXiv:2201.01236v1 [math.CT] (2022).

[4] J. Lurie, Higher Algebra, http://www.math.harvard.edu/~lurie/papers/HA.pdf