INTRODUCTION TO FLOER HOMOLOGY

Recap on Morse theory:

- In Morse theory we start with a Riemannian manifold M and a function $f : M \to \mathbb{R}$ (satisfying Morse conditions, namely all critical points are non-degenerate and some spaces of flows arriving and leaving to/from critical points intersect transversally).
- Each critical point has an associated finite index, given by the number of negative eigenvalues of the Hessian at the point, and this index is taken as the grading of a chain complex spanned by the critical points.
- The differential of a critical point x is a sum of a signed count of the gradient flow lines from x to points of index $\mathrm{Ind}(x) - 1$.
- This defines a homology $H_*(M, f)$ which is isomorphic to singular homology.

Floer analogue:

- Consider contractible loops $\mathcal{L}(M)$ on a symplectic manifold M. This space is infinite-dimensional! So given a functional $f : \mathcal{L}(M) \to \mathbb{R}$ the index of a critical point is infinite. However, there is a relative notion of index $\mu(x, y) =$ "$\mathrm{Ind}(x) - \mathrm{Ind}(y)$".
- We need a metric on $\mathcal{L}(M)$, which is defined via almost complex structures compatible with w. This gives rise to a formal gradient.
- Solutions of the gradient equation, if they exist, may not converge to critical points, but this has a fix.
- ...

1. Hamiltonian preliminaries

Let (M, w) be a closed symplectic manifold. Since w is closed, it induces an isomorphism $I_w : T^*M \to TM$.

Definition 1.1. A *Hamiltonian vector field* $X_H : M \to TM$ is the image via I_w of an exact 1-form $dH : M \to T^*M$. It is thus determined by the condition $w(X_H, -) = -dH$ and we say that $H : M \to \mathbb{R}$ is the *generating Hamiltonian function*.

We will actually be considering time-dependent 1-periodic Hamiltonian functions $H : M \times \mathbb{R} \to M$, so $H_t = H_{t+1}$, where $H_t := H(-, t)$. Also, note that if we change H_t by a time-dependent constant, then X_H does not change. We will assume our Hamiltonian functions are *normalized*:

$$\int_M H_t \frac{w^n}{n!} = 0.$$

From now on, we fix such a normalized Hamiltonian function $H : M \times \mathbb{R}/\mathbb{Z} \to M$.

Definition 1.2. The *Hamiltonian flow* is a 1-parameter family of diffeomorphisms $\{\varphi_t^H : M \to M\}$ with $t \in [0, 1]$ such that $\varphi_0^H = 1$ and for all $p \in M$, the curve $\gamma(t) := \varphi_t(p)$ satisfies

$$\boxed{\gamma'(t) = X_{H_t}(\gamma(t))}.$$

Remark 1.3. There is a one to one correspondence between fixed points of φ_1^H and loops $\gamma : \mathbb{R}/\mathbb{Z} \to M$ satisfying the boxed equation. Denote by $P(H)$ this set of curves.

Definition 1.4. A loop $\gamma \in P(H)$ is *non-degenerate* if $\det(1 - d\varphi_1^H(\gamma(0))) \neq 0$.

Arnold's conjecture. Assume that every point in $P(H)$ is non-degenerate. Then the number of points in $P(H)$ is topologically bounded below:

$$\#P(H) \geq \sum \mathrm{dim} H^k(M;\mathbb{Q}).$$

- This was proved by Floer in the case $\pi_2(M) = 0$ / monotone symplectic manifolds. Then extended by others to ther general case, using the ideas of Floer. An intermediate recurrent condition is to ask that w and c_1 of TM vanish on π_2.
- Note that if $H_t = H$ is time-independent, then the condition that all points in $P(H)$ are non-degenerate implies that $H : M \to \mathbb{R}$ is a Morse function and Arnold's conjecture follows from Morse theory.

2. Symplectic action functional

Fix a symplextic manifold (M, w) with $\pi_2(M) = 0$ and a normalized Hamiltonian function H_t. Let

$$\mathcal{L}(M) := \{\gamma : S^1 \to M; \gamma \text{ contractible } \}.$$

For any $\gamma \in \mathcal{L}(M)$, since γ is contractible, we may find a *spanning disc*, i.e., a smooth map

$$v : D := \{z \in \mathbb{C}; |z| \leq 1\} \to M$$

such that $v(e^{2\pi it}) = \gamma(t)$.

Define the *symplectic action functional* by:

$$\boxed{\mathcal{A}_H(\gamma) := -\int_D v^*(w) + \int_0^1 H_t(\gamma(t))dt.}$$

- The above does not depend on the choice of v if and only if
$$\int_{S^2} g^* w = 0 \text{ for all } g : S^2 \to M,$$
so that w vanishes on $\pi_2(M)$. So the simplest is to ask for $\pi_2(M) = 0$ but there are many intermediate conditions that also work.
- Also, why this disc trick? Let's consider the case $M = \mathbb{R}^2$. Then $w = d\lambda$ with $\lambda = ydx$ and the first summand of $\mathcal{A}_H(\gamma)$ is
$$\int_{S^1} \lambda = \int_0^1 y(t)x'(t)dt.$$
but when w is not exact (and w is never exact if M is compact) this is not defined.
- Note that if the second summand is trivial, then the only critical points are constant loops. The perturbation by a Hamiltonian avoids this degeneracy.

Thinking à la Morse, we want to study the critical points of this functional.

Lemma 2.1. *A loop $\gamma \in \mathcal{L}(M)$ is a critical point of $\mathcal{A}_H$ if and only if $\gamma \in P(H)$.*

Proof. This is easily checked if we identify tangent vectors of $\mathcal{L}(M)$ at γ with time-dependent vector fields ξ of TM such that $\xi(t) \in T_{\gamma(t)}M$ and $\xi(t) = \xi(t+1)$. We get:

$$d\mathcal{A}_H(\gamma)[\xi] = \int_0^1 w\left(X_{H_t}(\gamma(t)) - \gamma'(t), \xi(t)\right) dt.$$

□

Remark 2.2. Note that the 1-form $d\mathcal{A}_H(\xi)$ is well defined also when γ is not contractible, and when w does not vanish on $\pi_2(M)$. Such a form is exact on the space of contractible loops if and only if w vanishes on $\pi_2(M)$.

To study the gradient flow lines of this functional we require a metric. We do this by taking a loop J_t of w-compatible almost complex structures ($J_t : TM \to TM$ satisfy $J_t^2 = -I$ and $w(-, J_t-)$ is a Riemannian metric). Then on $T\mathcal{L}(M)$ we obtain an inner product on each fiber of $T\mathcal{L}(M)$:

$$\langle \xi, \eta \rangle_\gamma := \int_0^1 w\left(\xi(t), J_t\eta(t)\right) dt \quad ; \quad \xi, \eta \in T_\gamma \mathcal{L}(M).$$

The construction of Floer homology relies on the study of the *gradient flow lines* of the symplectic functional. These are the "loops of loops" $\mathbb{R} \to \mathcal{L}(M)$, $s \mapsto u(s, -)$ which are solutions to the equation

$$\frac{\partial u}{\partial s} + \mathcal{G}\mathrm{rad}\mathcal{A}_H(u(s, -)) = 0.$$

Remark 2.3. Here $\mathcal{G}\mathrm{rad}\mathcal{A}_H$ is defined to be the tangent vector field along γ such that

$$\langle \mathcal{G}\mathrm{rad}\mathcal{A}_H, \xi \rangle_\gamma = d\mathcal{A}_H(\gamma)[\xi].$$

does not define a regular tangent vector field on the completion of the space of smooth loops with respect to our metric (it is not even well-defined on curves γ which are not differentiable), so we cannot expect it to define a gradient flow on such a space. In other words, the above evolution equation is not a well-posed Cauchy problem. However, it turns out to be a nice PDE.

We can view our solutions as cylinders $u : \mathbb{R} \times \mathbb{R} \to M$ such that $u(s, t) = u(s, t + 1)$ and

$$\boxed{\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0.}$$

Here the gradient ∇ is taken with respect to the inner product $\langle \cdot, \cdot \rangle_t$ on M, so $w(\nabla H_t, J_tY) = dH_t(Y)$.

(a) If J, H and u are all independent of t then we recover the gradient flow of the vector field $\nabla H : M \to TM$.
(b) If $u(s, t) = \gamma(t)$ is independet of s then we recover the Hamiltonian equation.
(c) If $H_t = 0$ and $J_t = J$ is constant, the above is the equation for J-holomorphic curves (with the cylinder as a domain. They can be considered on any Riemann surface). This equation is a perturbed version of Cauchy-Riemann equation. Idea of Gromov: can extend methods from holomorphic analysis to the almost complex/symplectic setting.

3. Conley-Zehnder index

The set of symplectomorphisms of $(\mathbb{R}^{2n}, w_0)$ is given by the *symplectic group* $\mathrm{Sp}(2n)$, defined by matrices A of size $2n \times 2n$ with real entries such that

$$A^t JA = J, \text{ with} J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

Let $\gamma \in P(H)$ be non-degenerate. This happens if and only if

$$d\varphi_1^H : T_{\gamma(0)}M \to T_{\gamma(0)}M$$

does not have 1 as an eigenvalue. Geometrically, non-degeneracy is equivalent to the graph of φ_1^H being transversal to the diagonal at (γ, γ).

Linearizing the flow φ_t^H at $\gamma(0)$ we obtain a linear symplectomorphism

$$d\varphi_t(\gamma(0)) : T_{\gamma(0)}M \to T_{\gamma(t)}M.$$

In order to obtain a path of symplectic matrices we trivialize:

$$\begin{array}{ccc} T_{\gamma(0)}M & \xrightarrow{d\varphi_t^H(\gamma(0))} & T_{\gamma(t)}M \\ \uparrow & & \uparrow \\ \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^{2n} \end{array}$$

This is done by choosing a disc spanning γ and noting that v^*TM is a symplectic vector bundle over a contractible base space, so $v^*TM \cong D \times (\mathbb{R}^{2n}, w_0)$. In this way we obtain a smooth path $\Phi : [0,1] \to \mathrm{Sp}(2n)$ such that $\Phi(0) = 1$ and $\Phi(1)$ does not have 1 in its eigenvalues.

The *Conley-Zehnder index* $\mu(\Phi)$ of Φ is an intersection number between Φ and the "cycle" $\Sigma \in \mathrm{Sp}(2n)$ consisting of all matrices A possessing 1 as eigenvalue.

...and we skip lots of details here...

Define the *index of* γ by letting

$$\mathrm{Ind}(\gamma) := n - \mu(\Phi).$$

4. Floer homology

The key observation by Floer: in order to build a Morse complex in analogy to the finite dimensional case, it is not necessary to have a globally defined gradient flow. It is enough to have a nice structure for the spaces of gradient flow lines connecting two critical points.

Given critical points $\gamma, \eta \in P(H)$, denote by $\widetilde{M}(\gamma, \eta)$ the space of gradient trajectories u from γ to η, so that:

$$\lim_{s\to\infty} u(s,t) = \eta(t) \text{ and } \lim_{s\to-\infty} u(s,t) = \gamma(t).$$

Consider the $\mathbb{R}$-action on $\widetilde{M}(\gamma, \eta)$ given by $T \cdot u(s,t) = u(s+T, t)$ and let $\mathcal{M}(\gamma, \eta) = \widetilde{M}(\gamma, \eta)/\mathbb{R}$.

Non-trivial fact: generically $\mathcal{M}(\gamma, \eta)$ is a compact finite dimensional manifold of dimension

$$\dim \mathcal{M} = Ind(\gamma) - Ind(\eta) - 1.$$

This uses Sobolev spaces+Fredholm operators etc.

When it is a finite collection of points, write $n(\gamma, \eta) = \#\mathcal{M}(\gamma, \eta) \bmod \mathbb{Z}_2$.

$$CF_k(M, H) := \mathbb{Z}_2\langle \gamma \in P; \mathrm{Ind}(\gamma) = k\rangle$$

The differential of this chain complex is defined by counting the function's gradient flow lines connecting certain pairs of critical points: let $\partial : CF_k \to CF_{k-1}$ be given by

$$\partial_k(\gamma) := \sum_{\mathrm{Ind}(\eta)=k-1} n(\gamma, \eta)\eta.$$

The fact that $\partial^2 = 0$ is again non-trivial and uses Gromov's Compactness Theorem for sequences of J-holomorphic curves.

Now $HF_k(M, H, J)$ is the homology of this chain complex. This actually depends only on $\varphi = \varphi_1^H$ (thanks to normalization of H's). It is also independent of J.

Remark 4.1. If $H : \mathbb{R} \to M$ is a C^∞-small Morse function then $P(H)$ are constant loops and the Floer complex is isomorphic to the Morse complex.

Remark 4.2. Floer homology computes the homology of the manifold! So Arnold's conjecture follows.