CLASSIFICATION OF PERSISTENT MODULES AND SOME GEOMETRIC EXAMPLES

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Summary of my talk introducing persistence modules and discussing a couple of examples arising form geometry. Mainly followed the first chapter of [PRSZ19].

1. Persistence modules

Let $\mathbf{k}$ be a field and denote by $\mathrm{Vect}_{\mathbf{k}}$ the category of finite-dimensional vector spaces over $\mathbf{k}$. We will consider the real numbers as a posetal category $(\mathbb{R}, \geq)$.

Definition 1.1. A *persistence module* is a functor $V : (\mathbb{R}, \geq) \longrightarrow \mathrm{Vect}_{\mathbf{k}}$. Therefore it is given by a family $\{V_t\}_{t\in\mathbb{R}}$ of finite dimensional vector spaces over $\mathbf{k}$ together with morphisms $\pi_{st} : V_s \to V_t$ for all $s \leq t$ such that the diagram

$$\begin{array}{ccc} V_s & \longrightarrow & V_t \\ & \searrow \quad \nearrow & \\ & V_r & \end{array}$$

commutes. A *morphism of persistence modules* is just a natural transformation. Therefore it is given by maps $f_t : V_t \to V'_t$ compatible with the maps $\pi_{s,t}$. This defines a category $\underline{\mathrm{Pers}}$ of persistence modules.

Remark 1.2. The category of persistence modules is abelian. In fact, many of the constructions for persistence modules are also valid if we replace $\mathrm{Vect}_{\mathbf{k}}$ by an abelian category. In particular, one can consider persistence modules with values in cochain complexes instead of vector spaces.

We will assume the following *finite type conditions*:

(a) For any $t \in \mathbb{R} \setminus S$, with S a finite set, there exists a neighborhood U of t such that $\pi_{s,r}$ is an isomorphism for all $s < r$ in U.
(b) $V_s = 0$ for s sufficiently small.

Note that (a) implies that there is a finite "number of jumps". In particular, for s sufficiently large, we have that $V_s = V_\infty$.

We will also consider the following *semi-continuity property*:

(c) For all $t \in \mathbb{R}$ and for all $s \leq t$ with $t - s$ sufficiently small, the map $\pi_{s,t}$ is an isomorphism.

This last condition is easily visualized in the following special persistent modules:

Definition 1.3. Let I denote an interval of the form $(a, b]$ or $(a, +\infty)$. Define the *interval module* $\mathbf{k}(I)$ by letting

$$\mathbf{k}(I)_t := \begin{cases} \mathbf{k}; & \text{if } t \in I \\ 0; & \text{otherwise} \end{cases} ; \pi_{st} := \begin{cases} 1; & \text{if } s, t \in I \\ 0; & \text{otherwise} \end{cases} .$$

Theorem 1.4 (Normal forms). *Every persistent module V is isomorphic to a direct sum of interval modules*

$$V \cong \bigoplus_{i=1}^{N} \mathbf{k}(I_i)_i^m$$

with $I_i \neq I_j$ for all $i \neq j$. This isomorphism is unique up to permutations.

Proof. We may define a functor from persistent modules to the category of $\mathbf{k}[t]$-graded modules by sending V to the graded module $M = \bigoplus_t V_t$ together with the action

$$t \cdot (v_0, v_1, \cdots) := (0, \pi_{01}(v_0), \pi_{12}(v_1), \cdots).$$

Note that conditions $(a) - (c)$ ensure that M is finitely generated. This functor is an equivalence of categories, with obvious inverse. We may now apply the Structure Theorem for PID's to obtain isomorphisms

$$M \cong \bigoplus_{i=0}^{n} T^{\alpha_i}\mathbf{k}[t] \oplus \bigoplus T^{\gamma_j}\mathbf{k}[t]/(t^{n_j}).$$

This proves the existence of decompositions, where the intervals I_i are given by $(\alpha_i, +\infty)$ and $(\gamma_j, \gamma_j + n_j)$.

To prove uniqueness, note that $\mathrm{End}(\mathbf{k}[I]) \cong \mathbf{k}$. Indeed, any endomorphism of $\mathbf{k}[I]_t$ is given by multiplication by a certain λ_t. The compatibility with the morphisms $\pi_{s,t} = 1$ give $\lambda_s = \lambda_t$ for all $s, t \in I$. Now, assume we have

$$\bigoplus \mathbf{k}(I_i) \cong V \cong V' \cong \bigoplus \mathbf{k}(J_j).$$

Consider the compositions:

$$f_{ij} : K(I_i) \hookrightarrow V \cong V' \twoheadrightarrow K(Jj)$$

and

$$g_{ij} : K(J_j) \hookrightarrow V' \cong V \twoheadrightarrow K(I_i).$$

Then we have $\sum g_{ij} f_{ij} = 1$ and so at least one component is non-zero. For this component we get isomorphisms of the corresponding interval modules, so we may proceed inductively. □

The above result tells us that every persistent modules has a uniquely defined barcode:

Definition 1.5. The *barcode* associated to a persistent module is the collection of interval modules together with their multiplicities, given by the above decomposition:

$$\mathcal{B}(V) := \{(I_i, m_i)\}.$$

Definition 1.6. A point $t \in \mathbb{R}$ is said to be *spectral* if for any neighborhood U of t there is $s < r$ in U such that $^o pi_{sr}$ is not an isomorphism. We define the finite set

$$\mathrm{Spec}(V) := \{ \text{ spectral points } \} \cup \{+\infty\}.$$

This set is an isomorphism invariant of V.

2. Interleaving distance

For $\delta \geq 0$, denote by $T^\delta : (\mathbb{R}, \geq) \longrightarrow (\mathbb{R}, \geq)$ the translation functor $t \mapsto t + \delta$ and by $\eta^\delta : \mathrm{Id} \Rightarrow T^\delta$ the obvious natural transformation.

Given a persistence module V, we obtain a δ-translated persistent module by letting

$$V[\delta] := V \circ T^\delta.$$

Therefore we have $V[\delta]_t = V_{t+\delta}$ and $(\pi[\delta])_{s,t} = \pi_{s+\delta,t+\delta}$. The natural transformation η^δ induces a natural morphism $\phi^\delta : V \to V[\delta]$.

Definition 2.1. We say that two persistence modules V and V' are *δ-interleaved* if and only if there exist morphisms of persistence modules $F : V \to V'[\delta]$ and $G : V' \to V[\delta]$ such that the following diagrams commute:

$$\begin{array}{ccc} V & \xrightarrow{F} & V'[\delta] \\ & \searrow^{\phi^{2\delta}} & \downarrow{\scriptstyle G[\delta]} \\ & & V[2\delta] \end{array} \quad ; \quad \begin{array}{ccc} V' & \xrightarrow{G} & V[\delta] \\ & \searrow^{\phi^{2\delta}} & \downarrow{\scriptstyle F[\delta]} \\ & & V[2\delta] \end{array}$$

Definition 2.2. We define the *interleaving distance* between two persistence modules V and V' by letting

$$d_{int}(V, V') := \inf \{\delta \geq 0; V \text{ and } V' \text{ are } \delta\text{-interleaved } \}.$$

This is a pseudo-metric on the isomorphism classes of persistence modules with the same V_∞ (note that V and V' are δ-interleaved with $\delta < \infty$ if and only if $V_\infty = V'_\infty$). The semi-continuity condition (c) above ensures that this is actually a (non-degenerate) metric.

Remark 2.3. On the space of barcodes there is also a well-defined distance, called the *bottleneck distance*, and the assignment $V \mapsto \mathcal{B}(V)$ is actually an isometry, so that

$$d_{int}(V, V') = d_{bottleneck}(\mathcal{B}(V), \mathcal{B}(V')).$$

3. Morse persistence modules

Let M be a compact manifold without boundary and let $f : M \to \mathbb{R}$ be a Morse function. We take the uniform norm $||f|| := \max|f|$. We obtain a persistence module $V(f)$ by letting

$$V(f)_t := H_*(\{f < t\}; \mathbb{Z}_2).$$

For any $s < t$ denote by $i_{s,t} : \{f < s\} \to \{f < t\}$ the inclusion. We let $\pi_{s,t} := (i_{s,t})_*$.

Lemma 3.1. *Let $f, g : M \to \mathbb{R}$ be two Morse functions. Then $d_{int}(V(g), V(g)) \leq ||f - g||$.*

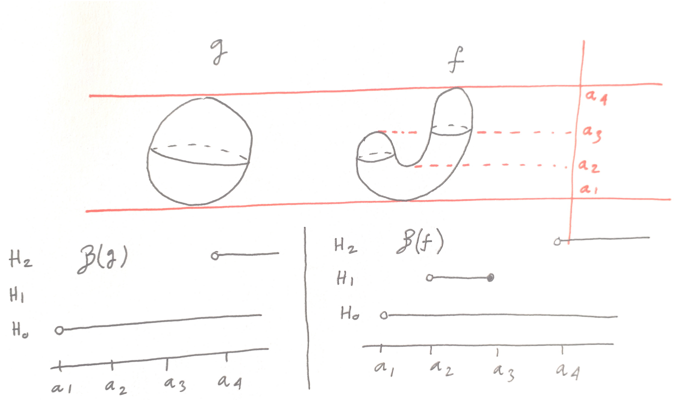
Proof. Note first that for any two Morse functions f, g with $f \leq g$ we have $\{g < t\} \hookrightarrow \{f < t\}$ and so we get a natural morphism of persistence modules $V(g) \to V(f)$.

Now, let $\delta : ||f - g||$. Then since $f - \delta \leq g$ we have a morphism $F : V(g) \to V(f - \delta) = V(f)[\delta]$. Also, since $g - \delta \leq f$ we get $G : V(f) \to V(g - \delta) = V(g)[\delta]$. Combining the inequalities $f - 2\delta \leq g - \delta \leq f$ we may complete these morphisms to the desired commutative diagrams, to see that $V(f)$ and $V(g)$ are δ-interleaved. □

Remark 3.2. Since for a diffeomorphism φ of M we have $V(f) \cong V(\varphi^* f)$, in fact, we have $d_{int}(V(g), V(g)) \leq \inf_{\varphi \in \mathrm{Diff}(M)} ||f - g||$.

The above lemma proves to be useful in order to quantify the obstructions for the approximation of Morse functions, as we will see in the following example.

Example 3.3. Let f be the height function associated to the deformed sphere shown below (which has 4 critical points) and consider the problem of approximating f by the height function g defined on the (round) sphere (with only two critical points).



The barcodes of f and g only differ in a finite interval $(a_2, a_3]$. By definition of the bottleneck distance we have

$$a_3 - a_2 = 2d_{bottleneck}(\mathcal{B}(f), \mathcal{B}(g)) = 2d_{int}(V(f), V(g)) \leq 2||f - g||.$$

therefore this gives

$$||f - g|| \geq \frac{1}{2}(a_3 - a_2).$$

This tells us that any perturbation g of f will be allowed whenever we either add or remove bars of lenght $\leq 2||f - g||$ or we extend/shorten bars from above and/or below by $||f - g||$.

4. Proper persistence modules and loop spaces

Let M a compact manifold without bundary and let g be a Riemannian metric on M. Given $t \in \mathbb{R}$ let $\Omega^t(M)$ denote the space of smooth loops $\gamma : S^1 \to M$ of length $< e^t$. Define a persistence module by letting $V(g)_t := H_*(\Omega^t M)$.

This is actually a *proper persistence module*, in which the finite-type condition (a) is replaced by:

(a') Spec(V) is a closed, discrete, bounded below subset of $\mathbb{R}$ (not necessarily finite).

Also, we remove condition (b). The new conditions also allow for a normal forms theorem and an isometry result. In terms of barcodes, the new conditions ensure there are finite numbers of barcodes of type $(-\infty, \infty)$, $(-\infty, b]$ ans $(a, b]$.

Now, let g' be another Riemannian metric on M. Since there is a constant C such that $C^{-1}g \leq g' \leq Cg$ we have

$$d_{int}(V(g), V(g')) \leq \frac{1}{2}\log C.$$

It follows that the equivalence class of the barcode of $V(g)$ is a topological invariant of M. Note that bars arising from $-\infty$ detect the homology of M, while bars going to $+\infty$ detect the homology of ΩM. The interesting information is contained in the finite bars, which account for contractions of loops modulo nullhomotopy. This has applications to the variational theory of geodesics (see [Wei19]).

References

[PRSZ19] L. Polterovich, D. Rosen, K. Samvelyan, and J. Zhang, *Topological persistence in geometry and analysis*, arXiv:1904.04044 (2019).

[Wei19] S. Weinberger, *Interpolation, the rudimentary geometry of spaces of Lipschitz functions, and geometric complexity*, Found. Comput. Math. **19** (2019), no. 5, 991–1011.