Set-Theoretical Background

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Our set-theoretical framework will be the Zermelo–Fraenkel axioms with the axiom of choice (ZFC):

- *Axiom of Extensionality.* If X and Y have the same elements, then $X = Y$.
- *Axiom of Pairing.* For all a and b there exists a set $\{a, b\}$ that contains exactly a and b.
- *Axiom Schema of Separation.* If P is a property (with a parameter p), then for all X and p there exists a set $Y = \{x \in X : P(x, p)\}$ that contains all those $x \in X$ that have the property P.
- *Axiom of Union.* For any X there exists a set $Y = \bigcup X$, the union of all elements of X.
- *Axiom of Power Set.* For any X there exists a set $Y = P(X)$, the set of all subsets of X.
- *Axiom of Infinity.* There exists an infinite set.
- *Axiom Schema of Replacement.* If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.
- *Axiom of Regularity.* Every nonempty set has a minimal element for the membership relation.
- *Axiom of Choice.* Every family of nonempty sets has a choice function.

1.1 Ordinals and cardinals

A set X is *well ordered* if it is equipped with a total order relation such that every nonempty subset $S \subseteq X$ has a smallest element. The statement that every set admits a well ordering is equivalent to the axiom of choice.

A set X is *transitive* if every element of an element of X is an element of X. An *ordinal* is a transitive set that is totally ordered with the membership relation (hence well ordered, by the axiom of regularity).

The first ordinals are $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, etc. Each ordinal is the well-ordered set of all smaller ordinals; that is,

$$\alpha = \{\beta : \beta < \alpha\}.$$

The first infinite ordinal is denoted by ω and it is equal to the set $\mathbb{N}$ of natural numbers with its canonical well ordering. Every ordinal α has a *successor*, namely $\alpha + 1 = \alpha \cup \{\alpha\}$. An ordinal that is not a successor ordinal is a *limit ordinal.* If α is a limit ordinal, then $\alpha = \bigcup \alpha$.

Every well-ordered set admits an order-preserving bijection with a unique ordinal. The class Ord of all ordinals is not a set (if it were, then it would be an ordinal and thus an element of itself, which contradicts the axiom of regularity).

Two sets X and Y are *equipotent* if there exists a bijection $X \to Y$. A *cardinal* is an initial ordinal; that is, an ordinal α such that if $\beta < \alpha$ then α and β are not equipotent. The *cardinality* of a set X is the least ordinal α equipotent with X (hence a cardinal). The cardinality of X will be denoted by $|X|$. Thus $|X| = |Y|$ if and only if there is a bijective function between X and Y, and $|X| \leq |Y|$ if and only if there is an injective function $X \to Y$. The Cantor–Bernstein Theorem asserts that if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$.

Cardinal arithmetic is defined as follows. If $|A| = \kappa$ and $|B| = \lambda$, then

$$\kappa + \lambda = |A \cup B|; \qquad \kappa \cdot \lambda = |A \times B|; \qquad \kappa^\lambda = |A^B|.$$

If $|A| = \kappa$, then $|P(A)| = 2^\kappa$. Cantor's Theorem states that $\kappa < 2^\kappa$ for all κ.

The first infinite cardinal is the ordinal ω. When viewed as a cardinal, it will be denoted by $\aleph_0$. For every cardinal κ, the *successor cardinal* κ^+ is the smallest cardinal which is greater than κ. For each ordinal α we denote $\aleph_{\alpha+1} = \aleph_\alpha^+$. If λ is a limit ordinal, then $\aleph_\lambda = \sup\{\aleph_\alpha : \alpha < \lambda\}$; then $\aleph_\lambda$ is called a *limit cardinal*.

The *generalized continuum hypothesis* (GCH) states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α. This statement is independent of the ZFC axioms. A special case is the equality $\aleph_1 = |\mathbb{R}|$.

1.2 Regular cardinals

A *sequence* is a function whose domain is an ordinal. A sequence with domain α will be denoted as $\langle x_i : i < \alpha \rangle$, and called an *$\alpha$-sequence*. The *limit* of a nondecreasing sequence of ordinals $\langle \gamma_i : i < \alpha \rangle$ is the supremum of the set $\{\gamma_i : i < \alpha\}$. A sequence of ordinals $\langle \gamma_i : i < \alpha \rangle$ is *continuous* if $\gamma_\beta = \lim_{i<\beta} \gamma_i$ for every limit ordinal $\beta < \alpha$.

An infinite cardinal κ is *regular* if it cannot be expressed as the sum of a smaller collection of smaller cardinals, and it is *singular* if it is not regular. Thus, κ is singular if and only if there exists a cardinal $\lambda < \kappa$ and a set $S = \{S_i : i < \lambda\}$ of subsets of κ such that $|S_i| < \kappa$ for all i, and $\kappa = \bigcup S$.

Equivalently, a cardinal κ is regular if it is equal to its own cofinality, where the *cofinality* of a limit ordinal α is the least limit ordinal β such that there is an increasing β-sequence of ordinals $\langle \gamma_i : i < \beta \rangle$ with $\lim \gamma_i = \alpha$. The cofinality of α is denoted by $\mathrm{cf}(\alpha)$. We have $\mathrm{cf}(\alpha) \leq \alpha$ for all α, and $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$ for all α.

The first infinite ordinal $\aleph_0$ is regular, and every successor cardinal $\aleph_{\alpha+1}$ is regular. For each ordinal α, the cardinal $\aleph_{\alpha+\omega}$ is singular with cofinality ω.

As an enlightening example, we are going to prove that, if λ is a regular cardinal, then the sets with less than λ elements are precisely the λ-presentable objects in the category of sets. Thus suppose that $|X| < \lambda$ and suppose given a function $f\colon X \to Y$ where $Y = \mathrm{colim}_{i<\lambda} Y_i$. Then each $x \in X$ is mapped into $f(x) \in Y_{i(x)}$ for some $i(x) < \lambda$. Therefore f factors through Y_k where $k = \sup\{i(x) : x \in X\}$. Since λ is regular, we may infer that $k < \lambda$, as k is a union of less than λ ordinals each of which is smaller than λ. Conversely, if X is λ-presentable then $|X| < \lambda$ since the identity $X \to X$ does not factor through any proper subset.

In Quillen's "small object argument", one starts with a simplicial set X of cardinality $\kappa = |X|$ and uses transfinite induction to construct a map $X \to Y$ by means of a λ-sequence $\langle X_i : i < \lambda \rangle$ with $X_0 = X$ and $Y = \operatorname{colim}_{i<\lambda} X_i$, where $\lambda = \kappa^+$ (which is regular since it is a successor cardinal).

1.3 Inaccessible cardinals

An uncountable limit cardinal that is regular is called *weakly inaccessible*.

A weakly inaccessible cardinal κ is *strongly inaccessible* if $\alpha < \kappa$ implies $2^\alpha < \kappa$. Hence, strongly inaccessible cardinals cannot be obtained from smaller cardinals by the operations of cardinal arithmetic. From now on, "innaccessible" will mean strongly inaccessible.

The existence of inaccessible cardinals cannot be proved in ZFC. Their existence is one of the so-called *large-cardinal axioms*.

The existence of arbitrarily large inaccessible cardinals is equivalent to the *universe axiom* of Grothendieck–Verdier, stating that every set is an element of some *Grothendieck universe*, that is, a transitive set U with $\omega \in U$ such that if x and y are in U then $\{x, y\} \in U$; if $x \in U$ then $P(x) \in U$; and every union of elements of U indexed by an element of U is in U.

1.4 The language of set theory

For a regular cardinal λ, a *λ-ary S-sorted signature* Σ consists of a set S of *sorts*, a set Σ_{op} of *operation symbols*, another set Σ_{rel} of *relation symbols*, and an *arity* function that assigns to each operation symbol an ordinal $\alpha < \lambda$, a sequence $\langle s_i : i < \alpha \rangle$ of *input sorts* and an *output sort* $s \in S$, and to each relation symbol an ordinal $\beta < \lambda$ and a sequence of sorts $\langle s_j : j < \beta \rangle$. An operation symbol with $\alpha = \emptyset$ is called *constant*. A signature Σ is called *operational* if $\Sigma_{\mathrm{rel}} = \emptyset$ and *relational* if $\Sigma_{\mathrm{op}} = \emptyset$.

Given an S-sorted signature Σ, a Σ-*structure* is a triple

$$X = \langle \{X_s : s \in S\},\ \{\sigma_X : \sigma \in \Sigma_{\mathrm{op}}\},\ \{\rho_X : \rho \in \Sigma_{\mathrm{rel}}\} \rangle$$

consisting of an *underlying S-sorted set*, denoted by $\{X_s : s \in S\}$ or $(X_s)_{s \in S}$, together with a function

$$\sigma_X \colon \prod_{i \in \alpha} X_{s_i} \longrightarrow X_s$$

for each operation symbol $\sigma \in \Sigma_{\mathrm{op}}$ of arity $\langle s_i : i < \alpha \rangle \to s$ (including a distinguished element of X_s for each constant symbol of sort s), and a set

$$\rho_X \subseteq \prod_{j < \beta} X_{s_j}$$

for each relation symbol $\rho \in \Sigma_{\mathrm{rel}}$ of arity $\langle s_j : j < \beta \rangle$.

Given a λ-ary S-sorted signature Σ, the *language* $\mathcal{L}_\lambda(\Sigma)$ consists of sets of *variables*, *terms*, and *formulas*, which are defined as follows. There is a family $W = \{W_s : s \in S\}$ of sets of cardinality λ, the elements of W_s being *variables* of sort s. One defines *terms* by declaring that each variable is a term and, for each operation symbol $\sigma \in \Sigma_{\text{op}}$ of arity $\langle s_i : i < \alpha \rangle \to s$ and each collection of terms τ_i of sort s_i, the expression $\sigma(\tau_i)_{i\in\alpha}$ is a term of sort s. *Atomic formulas* are expressions of the form $\tau_1 = \tau_2$ and $\rho(\tau_j)_{j\in\beta}$, where $\rho \in \Sigma_{\text{rel}}$ is a relation symbol of arity $\langle s_j : j < \beta \rangle$ and each τ_j is a term of sort s_j with $j \in \beta$. *Formulas* are built in finitely many steps from the atomic formulas by means of logical connectives and quantifiers. Thus, if $\{\varphi_i : i \in I\}$ are formulas and $|I| < \lambda$, then so are the conjunction $\bigwedge_{i\in I} \varphi_i$ and the disjunction $\bigvee_{i\in I} \varphi_i$. Quantification is allowed over sets of variables of cardinality smaller than λ; that is, $(\forall (x_i)_{i\in I})\, \varphi$ and $(\exists (x_i)_{i\in I})\, \varphi$ are formulas if φ is a formula and $|I| < \lambda$.

Variables that appear unquantified in a formula are called *free*. If a formula is denoted by $\varphi(x_i)_{i\in I}$, it is meant that each x_i is a free variable.

Each language $\mathcal{L}_\lambda(\Sigma)$ determines a *satisfaction relation* between Σ-structures and formulas with an assignment for their free variables. If $\varphi(x_i)_{i\in I}$ is a formula where each x_i is a free variable of sort s_i and X is a Σ-structure, a *variable assignment*, denoted by $x_i \mapsto a_i$, is a function $a\colon I \to \cup_{s\in S} X_s$ such that $a(i) \in X_{s_i}$ for all i. Satisfaction of a formula φ in a Σ-structure X is defined inductively, starting with the atomic formulas and quantifying over subsets of $\cup_{s\in S} X_s$ of cardinality smaller than λ. We write $X \models \varphi(a_i)_{i\in I}$ if φ is satisfied in X under an assignment $x_i \mapsto a_i$ for all its free variables x_i.

A formula without free variables is called a *sentence*. A set of sentences is called a *theory*. A *model* of a theory T in a language $\mathcal{L}_\lambda(\Sigma)$ is a Σ-structure satisfying all sentences of T.

A language $\mathcal{L}_\lambda(\Sigma)$ is called *finitary* if $\lambda = \omega$; otherwise it is *infinitary*. An especially important finitary language is the *language of set theory*. This is the finitary language corresponding to the signature with one sort, namely "sets", and one binary relation symbol ("membership"). Hence the atomic formulas are $x = y$ and $x \in y$, where x and y are sets.

1.5 Classes

A *class* consists of all sets for which a certain formula of the language of set theory is satisfied, possibly with parameters. More precisely, a class $\mathcal{C}$ is defined by a formula $\varphi(x, p)$ with a set of *parameters* p if

$$\mathcal{C} = \{x : \varphi(x, p)\}.$$

A class which is not a set is called a *proper class*. Each set A is definable with A itself as a parameter by $A = \{x : x \in A\}$.

A *model of ZFC* is a pair $\langle \mathcal{M}, \in \rangle$ where $\mathcal{M}$ is a class and $\in$ is the restriction of the membership relation to $\mathcal{M}$, in which the formalized ZFC axioms are satisfied. Thus, if we neglect the fact that $\mathcal{M}$ can be a proper class, we may view $\langle \mathcal{M}, \in \rangle$ as a Σ-structure where Σ is the relational signature of the language of set theory, and in fact a model of the theory consisting of the formalized ZFC axioms.

1.6 Absolute formulas

Given two classes $\mathcal{M} \subseteq \mathcal{N}$, we say that a formula $\varphi(x_1, \dots, x_k)$ of the language of set theory is *absolute between* $\mathcal{M}$ *and* $\mathcal{N}$ if, for all $a_1, \dots, a_k$ in $\mathcal{M}$,

$$\mathcal{N} \models \varphi(a_1, \dots, a_k) \text{ if and only if } \mathcal{M} \models \varphi(a_1, \dots, a_k).$$

A formula $\varphi(x_1, \dots, x_k)$ is *upward absolute* for models of some theory T if, given any two such models $\mathcal{M} \subseteq \mathcal{N}$ and given $a_1, \dots, a_k \in \mathcal{M}$ for which $\varphi(a_1, \dots, a_k)$ is true in $\mathcal{M}$, $\varphi(a_1, \dots, a_k)$ is also true in $\mathcal{N}$. We say that φ is *downward absolute* if, in the same situation, if $\varphi(a_1, \dots, a_k)$ holds in $\mathcal{N}$ then it holds in $\mathcal{M}$. A formula is *absolute* if it is both upward and downward absolute. If T is unspecified, then it should be understood that T is by default the set of all formalized ZFC axioms.

1.7 The cumulative hierarchy

Define, recursively on the class of ordinals,

$$V_0 = \emptyset, \qquad V_{\alpha+1} = P(V_\alpha)$$

for all α, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ if λ is a limit ordinal. It follows from the regularity axiom that every set is an element of some V_α. The *rank* of a set X is the least ordinal α such that $X \subseteq V_\alpha$. Hence

$$V_\alpha = \{X \mid \operatorname{rank} X < \alpha\}.$$

The *universe* $\mathcal{V}$ of all sets is the union of V_α for all ordinals α. Then $\langle \mathcal{V}, \in \rangle$ is a model of ZFC, and so is $\langle V_\kappa, \in \rangle$ for every inaccessible cardinal κ. If κ is inaccessible, then V_κ is a Grothendieck universe.

For a cardinal λ, we denote by $H(\lambda)$ the set of all sets whose transitive closure has cardinality less than λ. Thus $H(\lambda)$ is a transitive set contained in V_λ, and, if λ is inaccessible, then $H(\lambda) = V_\lambda$.

1.8 Defining categories

Let κ be an innaccessible cardinal. A *κ-category* is a quadruple $\mathcal{C} = (O, M, C, I)$ of sets in $V_{\kappa+1}$, where we denote $O = \operatorname{Ob}\mathcal{C}$ and call it the set of *objects* of $\mathcal{C}$ (hence each $X \in \operatorname{Ob}\mathcal{C}$ is in V_κ), and $M = \operatorname{Mor}\mathcal{C}$ is a collection of sets $\mathcal{C}(X, Y) \in V_\kappa$ for all $(X, Y) \in \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C}$, called *morphisms* from X to Y and equipped with a set C of associative *composition* functions

$$\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \longrightarrow \mathcal{C}(X, Z),$$

and I is a set consisting of a left and right identity $\mathrm{id}_X \in \mathcal{C}(X, X)$ for all $X \in \operatorname{Ob}\mathcal{C}$.

A κ-category $\mathcal{C}$ is called *small* (or *κ-small*) if $\mathcal{C} \in V_\kappa$. This is equivalent to imposing that $\operatorname{Ob}\mathcal{C}$ is in V_κ.

A *bundle of categories* $\mathcal{C}_*$ is a function that assigns a κ-category $\mathcal{C}_\kappa$ to every inaccessible cardinal κ. For example, if $\mathcal{C}$ is a λ-category, then

$$\mathcal{C}_\kappa = \begin{cases} \mathcal{C} & \text{if } \kappa > \lambda, \\ \emptyset & \text{otherwise.} \end{cases}$$

is a bundle of categories.

For example, if Set_κ denotes the category of sets of rank less than κ with their functions, then Set_* is a bundle of categories.

A *definable category* (or just a *category*) is a class

$$\mathcal{C} = \{X : \phi(X, p)\}$$

where ϕ is a formula in four free variables with a set of parameters p such that, for all inaccessible cardinals κ,

$$V_{\kappa+1} \models \phi(O, M, C, I, p)$$

if and only if (O, M, C, I) is a κ-category.

If $\mathcal{C}$ is a category defined by a formula $\phi(X, p)$, then

$$\mathcal{C}_\kappa = \{(O, M, C, I) \in V_{\kappa+1} \ : \ V_{\kappa+1} \models \phi(O, M, C, I, p)\}$$

is a bundle of categories.

For example, if R is a ring with 1, then the category of R-modules is defined by a formula ϕ with R as a parameter stating what is an R-module and what are R-module homomorphisms. More generally, for every theory T with signature Σ, the category of models of T (i.e., Σ-structures satisfying the sentences in T) is defined by a formula with parameters Σ and T.

A category is called *absolute* if it is definable by an absolute formula. If Σ is a λ-ary signature and T is a set of sentences in the language of Σ, then the categories of Σ-structures and models of T are absolute between transitive classes closed under sequences of length less than λ and containing Σ and T.

If $\mathcal{C}$ is a λ-category, then the formula

$$\exists\lambda \wedge (X \in \mathcal{C})$$

defines $\mathcal{C}$, where λ and $\mathcal{C}$ are parameters. The associated bundle is $\mathcal{C}_\kappa = \mathcal{C}$ if $\kappa > \lambda$ and $\mathcal{C}_\kappa = \emptyset$ otherwise.

1.9 Change of universe

Let $\mathcal{C}_*$ be a bundle of categories and let $\kappa < \kappa'$ be inaccessible cardinals. A *change of universe* of $\mathcal{C}_*$ from κ to κ' is a faithful functor $F\colon \mathcal{C}_\kappa \to \mathcal{C}_{\kappa'}$ that is injective on objects.

If $\mathcal{C}_*$ is definable by an upward absolute formula ϕ, then there is a canonical embedding $\mathcal{C}_\kappa \hookrightarrow \mathcal{C}_{\kappa'}$ for all $\kappa < \kappa'$. The κ'-category $\mathcal{C}_{\kappa'}$ is called the *logical enlargement* of $\mathcal{C}_\kappa$.

Let κ be an inaccessible cardinal, and let λ be a regular cardinal with $\lambda < \kappa$. For a small κ-category B, let $\mathrm{Cont}_\lambda(B, \mathrm{Set}_\kappa)$ denote the full subcategory of $\mathrm{Fun}(B, \mathrm{Set}_\kappa)$ of λ-*continuous* functors, i.e., functors $B \to \mathrm{Set}_\kappa$ that preserve all λ-small limits that exist in B.

If A is a small κ-category, then the Yoneda embedding factors as

$$A \hookrightarrow \mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_\kappa) \hookrightarrow \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Set}_\kappa).$$

While $A \hookrightarrow \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Set}_\kappa)$ is a *free cocompletion*, we say that $A \hookrightarrow \mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_\kappa)$ is a λ-*free cocompletion* of A, meaning that it preserves all λ-small colimits that exist in A and every functor $A \to \mathcal{C}$ with $\mathcal{C}$ cocomplete that preserves λ-small colimits extends uniquely to a colimit-preserving functor $\mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_\kappa) \to \mathcal{C}$.

The Representation Theorem states that a κ-category is locally λ-presentable if and only if it is equivalent to $\mathrm{Cont}_\lambda(B, \mathrm{Set}_\kappa)$ for some small κ-category B.

Let $\mathcal{C}$ be a locally λ-presentable κ-category, where $\lambda < \kappa$. Let A be a skeleton of the full subcategory of λ-presentable objects in $\mathcal{C}$. Then $\mathcal{C} \simeq \mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_\kappa)$.

If $\kappa' > \kappa$, the κ'-category $\mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_{\kappa'})$ is called the *locally presentable enlargement* of $\mathcal{C}$ from κ to κ'.

By the Representation Theorem, $\mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_{\kappa'})$ is locally λ-presentable. Furthermore, the embedding

$$\mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_\kappa) \hookrightarrow \mathrm{Cont}_\lambda(A^{\mathrm{op}}, \mathrm{Set}_{\kappa'})$$

preserves colimits (and limits), since they are calculated in sets.

For example, the category Gr of groups can be defined in at least two (equivalent) ways:

(a) By means of a formula ϕ stating what is a group and what are group homomorphisms.

(b) By means of a formula defining $\mathrm{Cont}_{\aleph_0}(A^{\mathrm{op}}, \mathrm{Set})$ where A is a skeleton of the full subcategory of finitely presentable groups.

Since the formula ϕ is absolute, the logical enlargement $\mathrm{Gr}_\kappa \hookrightarrow \mathrm{Gr}_{\kappa'}$ exists. Since this embedding coincides with the locally presentable enlargement, it preserves limits and colimits.

The same happens for all categories of structures or categories of models of theories. However, there is an ω-sequence of monads on Set_κ whose colimit depends on κ (Bowler, 2012).

It is also important to be aware of the fact that the property of being locally presentable need not be preserved under changes of universe. Thus, there are inaccessible cardinals $\kappa < \kappa'$ and a definable category $\mathcal{C}$ such that $\mathcal{C}_\kappa$ is locally presentable but $\mathcal{C}_{\kappa'}$ is not. The following is an example. Choose an inaccessible cardinal λ and a λ-category $\mathcal{A}$ that is not locally presentable. Let $\mathcal{C}$ be the category defined by the formula

$$\exists \lambda \wedge (X \in \mathcal{A})$$

Thus $\mathcal{C}_\kappa = \emptyset$ if $\kappa \leq \lambda$ while $\mathcal{C}_\kappa = \mathcal{A}$ if $\kappa > \lambda$.

1.10 Size conventions

Let κ be an inaccessible cardinal, which we assume given and will remain implicit in most statements. We view V_κ as a Grothendieck universe. Thus a set is *small* if it is κ-small, that is, if $X \in V_\kappa$. It is useful to call *large* those sets in $V_{\kappa+1} \smallsetminus V_\kappa$, and *very large* the ones beyond this. If $\mathcal{C}$ is a κ-category (so, by definition, $\mathcal{C} \in V_{\kappa+1}$) then the nerve $N\mathcal{C}$ is a possibly large simplicial set, i.e., $N\mathcal{C} \in V_{\kappa+1}$.

The condition that $\mathcal{C}(X,Y) \in V_\kappa$ for all X and Y is described by saying that $\mathcal{C}$ is *locally small*. Usually all categories are assumed to be locally small.

Certain constructions, such as passage to a category of fractions, lead to categories in a larger Grothendieck universe, that is, a set $V_{\kappa'}$ where κ' is inaccessible and $\kappa' > \kappa$.

A *quasicategory* is a simplicial set $X \in V_{\kappa+1}$ satisfying the weak Kan condition, and it is *small* if $X \in V_\kappa$. According to Lurie, a quasicategory X is *essentially small* if it satisfies the following equivalent conditions:

(a) The set of equivalence classes of objects of X is small, and for every morphism $f\colon x \to y$ in X the sets $\pi_0(\mathrm{map}(x,y))$ and $\pi_n(\mathrm{map}(x,y), f)$ are small for $n \geq 1$.

(b) There exists a small quasicategory A and an equivalence $A \to X$ of quasicategories.

(c) X is κ-compact as an object of the quasicategory Cat_∞ of quasicategories.

If X is a Kan complex, then it is essentially small as a quasicategory if and only if $\pi_0(X)$ is small and $\pi_n(X,x)$ is small for $n > 0$ and every vertex $x \in X$. This condition is equivalent to the statement that there exists a weak homotopy equivalence $K \to X$ where K is a small simplicial set, and also to the claim that X is κ-compact when regarded as an object of the quasicategory $\mathbf{S}$ of spaces.

A quasicategory X is *locally small* if $\mathrm{map}(x,y)$ is weakly equivalent to a small simplicial set for all $x, y \in X$. This is equivalent to the statement that for every small set S of objects of X the full subcategory spanned by S is essentially small.

If A is essentially small and X is locally small, then $\mathrm{Fun}(A,X)$ is locally small.

The quasicategory Cat_∞ of all quasicategories is in $V_{\kappa'+1}$ where κ' is inaccessible and $\kappa' > \kappa$.

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