Geometric Uses of Persistent Homology

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Stability Results for Point Clouds

A *point cloud* is an unordered finite collection $X = \{x_0, \dots, x_n\}$ of points in $\mathbb{R}^N$ for some $N \geq 2$. We view X as a finite metric space with the Euclidean distance.

1 Čech complexes and Vietoris–Rips complexes

For each real number $\varepsilon \geq 0$, the *Čech complex* $C_\varepsilon(X)$ of a point cloud X is the abstract simplicial complex with vertex set X whose k-faces are collections of points $\{x_{i_0}, \dots, x_{i_k}\}$ such that the closed balls of radius $\varepsilon/2$ centered at them have at least one point of common intersection:

$$\bar{B}_{\varepsilon/2}(x_{i_0}) \cap \cdots \cap \bar{B}_{\varepsilon/2}(x_{i_k}) \neq \emptyset.$$

Likewise, for each real number $\varepsilon \geq 0$, the *Vietoris–Rips complex* $R_\varepsilon(X)$ is the abstract simplicial complex with vertex set X whose k-faces are collections of points $\{x_{i_0}, \dots, x_{i_k}\}$ of diameter at most ε; that is, such that

$$d(x_{i_r}, x_{i_s}) \leq \varepsilon \text{ for all } r, s \in \{0, \dots, k\}.$$

For every X and every ε, there is an inclusion $C_\varepsilon(X) \subseteq R_\varepsilon(X)$. For sufficiently small values of ε, both complexes are equal and discrete (in bijection with X), while for large enough values of ε they are also equal and their geometric realization is a single n-simplex if X has cardinality $n + 1$.

A *flag complex* is an abstract simplicial complex where every collection of pairwise adjacent vertices spans a face. Thus every flag complex is maximal among those with a given 1-skeleton, and this means that it can be stored by listing only its edges. The Vietoris–Rips complex of every point cloud is a flag complex.

If $C_\varepsilon(X)$ is the Čech complex of a point cloud X, then the geometric realization $|C_\varepsilon(X)|$ is homotopy equivalent to the union of the closed balls of radius $\varepsilon/2$ centered at the points of X. This is an instance of the *Nerve Theorem*: If $\mathcal{U}$ is an open cover of a paracompact space K such that every nonempty intersection of finitely many sets in $\mathcal{U}$ is contractible, then K is homotopy equivalent to the nerve of $\mathcal{U}$. For a proof, see [5, Corollary 4.G.3].

2 Persistence modules

Fix any field $\mathbb{F}$. A *persistence module* over $\mathbb{F}$ is a functor from the real numbers $\mathbb{R}$ viewed as an ordered set to the category of $\mathbb{F}$-vector spaces of finite dimension. We can denote a persistence module as a pair (V, π) where $V = \{V_t\}_{t\in\mathbb{R}}$ is a collection of

$\mathbb{F}$-vector spaces of finite dimension and π is a collection of $\mathbb{F}$-linear maps $\pi_{s,t}\colon V_s \to V_t$ for $s \le t$ such that

$$\pi_{s,t} \circ \pi_{r,s} = \pi_{r,t} \text{ if } r \le s \le t, \text{ and } \pi_{t,t} = \mathrm{id} \text{ for all } t \in \mathbb{R}.$$

A persistence module is of *finite type* if there is a finite set $A = \{a_0, \ldots, a_k\} \subset \mathbb{R}$ with $a_0 < \cdots < a_k$ such that $V_t = 0$ for $t < a_0$, and

(i) for every $a \in A$ there is an $\varepsilon > 0$ such that $\pi_{a,t}$ is an isomorphism if $t \in [a, a+\varepsilon)$ and $\pi_{s,a}$ is not an isomorphism if $s \in (a - \varepsilon, a)$;

(ii) if $x \notin A$ then $\pi_{s,t}$ is an isomorphism for $s \le t$ in $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$.

The set A is called the *spectrum* of (V, π) and its elements are *spectral points*. We write V_∞ to denote the direct limit of (V, π) viewed as a directed diagram of $\mathbb{F}$-vector spaces. Thus, $V_\infty \cong V_t$ for $t \ge a_k$.

If X is a point cloud in $\mathbb{R}^N$ for some N and $R_t(X)$ is the Vietoris–Rips complex associated with X for each value of $t \ge 0$, while $R_t(X) = \emptyset$ if $t < 0$, then

$$V_t = H_*(R_t(X); \mathbb{F}) = \bigoplus_{i=0}^{\infty} H_i(R_t(X); \mathbb{F})$$

defines a persistence module of finite type, with $\pi_{s,t}$ the homomorphisms induced in homology by the inclusions $R_s(X) \subseteq R_t(X)$ if $s \le t$. This persistence module is called the *Vietoris–Rips module* of X. The *Čech module* of X is defined similarly using the Čech complex $C_t(X)$ for $t \ge 0$.

A *morphism* $f\colon (V, \pi) \to (V', \pi')$ of persistence modules over a field $\mathbb{F}$ is a natural transformation of functors; that is, a collection of $\mathbb{F}$-linear maps $f_t\colon V_t \to V'_t$ for $t \in \mathbb{R}$ such that

$$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$$

whenever $s \le t$. A morphism of persistence modules is an *isomorphism* if it has a two-sided inverse, that is, $g\colon (V', \pi') \to (V, \pi)$ with $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. Then it follows that f_t is an isomorphism for every t.

3 Normal form and barcodes

From now on we assume that all persistence modules are of finite type.

For every interval $I = [a, b) \subset \mathbb{R}$ with $a < b$ or $I = [a, \infty)$, define a persistence module $\mathbb{F}(I)$ as follows:

$$\mathbb{F}(I)_t = \begin{cases} \mathbb{F} & \text{if } t \in I \\ 0 & \text{otherwise,} \end{cases}$$

with $\pi_{s,t} = \mathrm{id}$ if $s, t \in I$ and $\pi_{s,t} = 0$ otherwise. Such persistence modules are called *interval modules*. Their spectrum is $\{a, b\}$ if $I = [a, b)$ or $\{a\}$ if $I = [a, \infty)$.

If (V, π) and (V', π') are persistence modules, their *direct sum* is the persistence module (W, θ) with $W_t = V_t \oplus V'_t$ for all t and $\theta_{s,t} = \pi_{s,t} \oplus \pi'_{s,t}$ for all s, t.

We denote, for every positive integer m,

$$\mathbb{F}(I)^m = \mathbb{F}(I) \oplus \overset{m}{\cdots} \oplus \mathbb{F}(I),$$

so $\mathbb{F}(I)^m$ also becomes a persistence module.

For every persistence module (V, π) there is a finite collection $\{I_1, \ldots, I_M\}$ with $I_i = [b_i, d_i)$ or $I_i = [b_i, \infty)$ for each i, such that $I_i \neq I_j$ if $i \neq j$, and there is an isomorphism of persistence modules

$$V \cong \bigoplus_{i=1}^{M} \mathbb{F}(I_i)^{m_i} \tag{3.1}$$

where $m_1, \ldots, m_M$ are positive integers. Moreover, the intervals I_i are unique up to their ordering. This is called the *Normal Form Theorem* and can be inferred from the classification of finitely generated graded modules over the polynomial ring $\mathbb{F}[t]$, which is a principal ideal domain, by letting $\mathbb{F}[t]$ act on the vector space $V_* = V_{a_0} \oplus \cdots \oplus V_{a_k}$ by

$$t \cdot v = \pi_{a_i, a_{i+1}}(v) \text{ if } v \in V_{a_i} \text{ with } i < k, \text{ and } t \cdot v = v \text{ if } v \in V_{a_k},$$

where $\{a_0, \ldots, a_k\}$ is the spectrum of (V, π) with $a_0 < \cdots < a_k$.

Hence we may represent each persistence module (V, π) by means of a *barcode* whose segments are the intervals $\{I_1, \ldots, I_M\}$ with multiplicities m_i given by (3.1).

4 Persistence diagrams

The *persistence diagram* for a persistence module (V, π) has a point (b_i, d_i) in a coordinate plane for each bounded interval $[b_i, d_i)$ in the normal form (3.1) of (V, π). Thus a point (b_i, d_i) in a persistence diagram represents a vector of V_* with *birth parameter* b_i and *death parameter* d_i.

The rays $[b_i, \infty)$ are represented as points (b_i, y_∞) where y_∞ is an arbitrary but fixed point above the largest value in the spectrum of (V, π). The multiplicites m_i are usually depicted by increasing the size of the corresponding dots in the picture.

It is also customary to include the diagonal $b = d$ in persistence diagrams, and view its points as having infinite multiplicity.

5 Interleaving distance

For a persistence module (V, π) and $\delta \in \mathbb{R}$, define another persistence module by

$$V[\delta]_t = V_{t+\delta} \text{ and } \pi[\delta]_{s,t} = \pi_{s+\delta,\, t+\delta}.$$

This is called a δ-*shift* of (V, π). We omit π from the notation from now on.

If $\delta \geq 0$, then there is a morphism of persistence modules $\sigma_\delta \colon V \to V[\delta]$ given by $(\sigma_\delta)_t = \pi_{t,\, t+\delta}$ for all $t \in \mathbb{R}$. Moreover, each morphism $f \colon V \to V'$ of persistence modules yields a morphism $f[\delta] \colon V[\delta] \to V'[\delta]$ for all $\delta \in \mathbb{R}$, namely $f[\delta]_t = f_{t+\delta}$ for all $t \in \mathbb{R}$.

For $\delta > 0$, two persistence modules V and V' are *δ-interleaved* if there exist morphisms $F\colon V \to V'[\delta]$ and $G\colon V' \to V[\delta]$ such that

$$G[\delta] \circ F = \sigma_{2\delta} \text{ and } F[\delta] \circ G = \sigma'_{2\delta}.$$

If V and V' are δ-interleaved for some $\delta > 0$, then $\dim_{\mathbb{F}} V_\infty = \dim_{\mathbb{F}} V'_\infty$ and hence $V_\infty \cong V'_\infty$. The *interleaving distance* between two persistence modules V and V' with $V_\infty \cong V'_\infty$ is defined as

$$d_{\text{int}}(V, V') = \inf\{\delta > 0 \mid V \text{ and } V' \text{ are } \delta\text{-interleaved}\}.$$

It follows that, if $a < b$ and $c < d$, then

$$\begin{aligned} &d_{\text{int}}(\mathbb{F}[a, b), \mathbb{F}[c, d)) \\ &\quad = \min\left\{\max\left\{\tfrac{1}{2}(b - a),\ \tfrac{1}{2}(d - c)\right\},\ \max\left\{|a - c|, |b - d|\right\}\right\} \end{aligned} \tag{5.1}$$

while

$$d_{\text{int}}(\mathbb{F}[a, \infty), \mathbb{F}[c, \infty)) = |a - c|. \tag{5.2}$$

From (5.1) and (5.2) it follows that the interleaving distance between two persistence modules V and V' with $V_\infty \cong V'_\infty$ is equal to the *bottleneck distance* between their persistence diagrams D and D', which is defined as follows.

A *matching* between D and D' is a correspondence (i.e., a subset of $D \times D'$, including their diagonals) where off-diagonal points appear precisely once. We view a matching as a function $\varphi\colon D \to D'$ and write

$$\|\varphi\| = \sup\{d_\infty((x, y),\ \varphi(x, y)) \mid (x, y) \in D\},$$

where d_∞ is the ℓ_∞-distance on $\mathbb{R}^2$, namely

$$d_\infty((x, y), (x', y')) = \max\{|x - x'|,\ |y - y'|\}.$$

Then the *bottleneck distance* between D and D' is defined as

$$W_\infty(D, D') = \inf\{\|\varphi\| \mid \varphi\colon D \to D'\},$$

where the infimum is taken over all matchings between D and D'.

The bottleneck distance is the case $p = \infty$, $q = \infty$ of the *Wasserstein distances*, defined for $p, q \geq 1$ as

$$W_p[q](D, D') = \inf_{\varphi : D \to D'} \left[\sum_{(x,y) \in D} d_q((x, y),\ \varphi(x, y))^p \right]^{1/p}$$

where d_q is the ℓ_q-distance on $\mathbb{R}^2$:

$$d_q((x, y), (x', y')) = \Big(|x - x'|^q + |y - y'|^q\Big)^{1/q}.$$

6 Hausdorff distance

Suppose that X and Y are nonempty subsets of a metric space M with distance d. For a point $x \in X$, define

$$d(x, Y) = \inf\{d(x, y) \mid y \in Y\},$$

and define also

$$d(X, Y) = \sup\{d(x, Y) \mid x \in X\}.$$

If X is bounded, then $d(X, Y)$ is finite; in fact, $0 \le d(X, Y) \le \operatorname{diam}(X) + d(x_0, y_0)$ for arbitrary points $x_0 \in X$ and $y_0 \in Y$. However, $d(X, Y) \neq d(Y, X)$ in general.

Now suppose that X and Y are compact. The *Hausdorff distance* between X and Y is defined as

$$d_H(X, Y) = \max\{d(X, Y),\, d(Y, X)\}.$$

Note that if $d(X, Y) = 0$ then $X \subseteq Y$, since Y is closed in M. Consequently, $d_H(X, Y) = 0$ if and only if $X = Y$. Moreover, d_H satisfies the triangle inequality and therefore d_H is indeed a distance on the set of nonempty compact subsets of M. However, if $M = \mathbb{R}$, $X = \mathbb{Q}$ and $Y = \mathbb{R} \smallsetminus \mathbb{Q}$ (not compact) then $d_H(X, Y) = 0$.

7 Gromov–Hausdorff distance

For X and Y nonempty compact metric spaces, the *Gromov–Hausdorff distance* between X and Y is defined as

$$d_{GH}(X, Y) = \inf\{d_H(f(X), g(Y)) \mid f\colon X \hookrightarrow M,\, g\colon Y \hookrightarrow M\},$$

where the infimum is taken over all isometric embeddings $f\colon X \hookrightarrow M$, $g\colon Y \hookrightarrow M$ into some common metric space M. Hence $d_{GH}(X, Y) = 0$ if and only if X and Y are isometric.

The Gromov–Hausdorff distance turns the set of isometry classes of compact metric spaces into a path-connected, complete, separable metric space. Convergence in this metric space is called *Gromov–Hausdorff convergence*; the source is [4].

An alternative description of the Gromov–Hausdorff distance is as follows. For nonempty compact metric spaces X and Y, a *surjective correspondence* between X and Y is a multivalued function from X to Y, that is, a subset $C \subseteq X \times Y$ such that for all $x_0 \in X$ there is some $(x_0, y) \in C$ and for all $y_0 \in Y$ there is some $(x, y_0) \in C$.

If C is a surjective correspondence between X and Y, then the inverse correspondence C^{-1} (that is, the set of points $(y, x) \in Y \times X$ for which $(x, y) \in C$) is also surjective.

The *distortion* of a correspondence $C \subseteq X \times Y$ is defined as

$$\operatorname{dis}(C) = \max\{|d_X(x, x') - d_Y(y, y')| \,:\, (x, y) \in C,\, (x', y') \in C\}.$$

For example, If $C = \{(x, f(x)) \mid x \in X\}$ for some function $f\colon X \to Y$, then $\operatorname{dis}(C) = 0$ if and only if f is an isometry.

The following result is proved in [6]:

$$d_{GH}(X,Y) = \frac{1}{2}\inf\{\mathrm{dis}(C) \mid C \subseteq X \times Y\},$$

where the infimum is taken over all surjective correspondences between X and Y. This makes the Gromov–Hausdorff distance between point clouds computable with a finite procedure, since $X \times Y$ is a finite set.

8 Stability for point clouds

In what follows, homology is meant with coefficients in any field $\mathbb{F}$, which is not specified. If f and g are simplicial maps from an abstract simplicial complex K to another abstract simplicial complex L, we say that f and g are *contiguous* if, for each face $\sigma = \{v_{i_0}, \ldots, v_{i_n}\}$ of K, the points

$$f(v_{i_0}), \ldots, f(v_{i_n}), g(v_{i_0}), \ldots, g(v_{i_n})$$

(which need not be distinct) form a face of L. Hence $f(\sigma)$ and $g(\sigma)$ are faces of L contained in some common larger face of L. From this fact it follows that, if f and g are contiguous, then they yield homotopic maps $|K| \to |L|$ on the geometric realizations of K and L; see [7, Theorem 12.5] for details. Consequently, they induce the same homomorphisms $H_n(K) \to H_n(L)$ for all n.

Suppose given two point clouds X and Y in $\mathbb{R}^N$ for some N. We consider their Vietoris–Rips complexes $R_t(X)$ and $R_t(Y)$ for each $t \geq 0$, and the corresponding persistence modules $V_t(X) = H_*(R_t(X))$ and $V_t(Y) = H_*(R_t(Y))$, where $R_t(X) = \emptyset$ and $R_t(Y) = \emptyset$ for $t < 0$.

The *Stability Theorem* states that

$$\boxed{\tfrac{1}{2}\, d_{\mathrm{int}}(V(X), V(Y)) \leq d_{GH}(X,Y)} \tag{8.1}$$

where d_{int} is the interleaving distance and d_{GH} is the Gromov–Hausdorff distance. The following proof of (8.1) has been extracted from the original source [1]. Essentially the same argument can be found in [8, Theorem 1.5.4].

In order to prove (8.1), we need to show that $V(X)$ and $V(Y)$ are δ-interleaved for $\delta = 2d_{GH}(X,Y)$.

Note that, since X and Y are finite sets, there is some correspondence $C \subseteq X \times Y$ with $\delta = \mathrm{dis}(C)$. A function $f\colon X \to Y$ is called *subordinate* to C if

$$\{(x, f(x)) \mid x \in X\} \subseteq C.$$

If $f\colon X \to Y$ is any function subordinate to C, then, for each face $\sigma = \{v_{i_0}, \ldots, v_{i_n}\}$ of $R_t(X)$, we have that $\mathrm{dis}(C) \geq |d(v_{i_k}, v_{i_\ell}) - d(f(v_{i_k}), f(v_{i_\ell}))|$ for all k and ℓ, from which we infer that

$$d(f(v_{i_k}), f(v_{i_\ell})) \leq d(v_{i_k}, v_{i_\ell}) + \mathrm{dis}(C) \leq t + \delta$$

for all k and ℓ, and this implies that $f(\sigma)$ is a face of $R_{t+\delta}(Y)$. Hence f induces a simplicial map $f_t\colon R_t(X) \to R_{t+\delta}(Y)$ for each $t \in \mathbb{R}$. Let

$$F_t\colon V_t(X) \longrightarrow V_{t+\delta}(Y) = V(Y)[\delta]_t$$

be the linear map induced by f_t in homology.

Similarly, we may choose any function $g\colon Y \to X$ subordinate to C^{-1}, and we obtain a simplicial map $g_t\colon R_t(Y) \to R_{t+\delta}(X)$ for every t and hence a linear map

$$G_t\colon V_t(Y) \longrightarrow V_{t+\delta}(X) = V(X)[\delta]_t$$

for each t. In fact the collections $\{F_t\}_{t\in\mathbb{R}}$ and $\{G_t\}_{t\in\mathbb{R}}$ define morphisms of persistent modules $F\colon V(X) \to V(Y)[\delta]$ and $G\colon V(Y) \to V(X)[\delta]$.

We next check that $V(X)$ and $V(Y)$ are δ-interleaved by means of F and G. The shift morphism

$$\sigma_{2\delta}\colon V(X) \longrightarrow V(X)[2\delta]$$

is induced in homology by the inclusions $R_t(X) \subseteq R_{t+2\delta}(X)$. Hence it is enough to prove that $(g \circ f)_t$ is contiguous to the inclusion map $R_t(X) \to R_{t+2\delta}(X)$ for each t. For this, pick any face $\{v_{i_0}, \ldots, v_{i_n}\}$ of $R_t(X)$. Then $(v_{i_\ell}, f(v_{i_\ell})) \in C$ and $(g(f(v_{i_k})), f(v_{i_k})) \in C$ for all k, ℓ, and this implies that

$$d(g(f(v_{i_k})), v_{i_\ell}) \le d(f(v_{i_k}), f(v_{i_\ell})) + \delta.$$

Next, we use that $(v_{i_k}, f(v_{i_k})) \in C$ and $(v_{i_\ell}, f(v_{i_\ell})) \in C$ to infer that

$$d(f(v_{i_k}), f(v_{i_\ell})) + \delta \le d(v_{i_k}, v_{i_\ell}) + 2\delta \le t + 2\delta.$$

Similarly, using that $(g(f(v_{i_k})), f(v_{i_k})) \in C$ and $(g(f(v_{i_\ell})), f(v_{i_\ell})) \in C$ for all k, ℓ, we obtain that

$$d(g(f(v_{i_k})), g(f(v_{i_\ell}))) \le d(f(v_{i_k}), f(v_{i_\ell})) + \delta \le d(v_{i_k}, v_{i_\ell}) + 2\delta \le t + 2\delta.$$

This proves that the points

$$g(f(v_{i_0})), \ldots, g(f(v_{i_n})), v_{i_0}, \ldots, v_{i_n}$$

form a face of $R_{t+2\delta}(X)$, as needed. The argument with $f \circ g$ is analogous. This concludes the proof of (8.1).

9 Stability for smooth functions

Let M be any topological space. For a function $f\colon M \to \mathbb{R}$ and any $t \in \mathbb{R}$, we may consider the *sublevel set*

$$L_t(f) = \{x \in M \mid f(x) \le t\} = f^{-1}(-\infty, t],$$

and denote

$$V_t(f) = H_*(L_t(f); \mathbb{R}),$$

where singular homology is meant here. Let $\pi_{s,t}\colon V_s(f) \to V_t(f)$ be induced by the inclusions $L_s(f) \subseteq L_t(f)$ if $s \le t$.

The function f is called *tame* if $V(f)$ is a persistence module (of finite type). This happens in the following cases, among other situations:

- If M is a closed interval $[a, b] \subset \mathbb{R}$, and f is a differentiable function with finitely many critical points.
- If M is a closed smooth manifold, and f is a Morse function, i.e., a smooth function with no degenerate critical points.
- If $M = \mathbb{R}^N$, and f is defined as

  $$f(p) = d(p, X),$$

  where X is a point cloud in $\mathbb{R}^N$.

The following inequality holds if f and g are tame functions $M \to \mathbb{R}$:

$$\boxed{d_{\text{int}}(V(f), V(g)) \leq \|f - g\|_\infty} \tag{9.1}$$

where $\|f - g\|_\infty = \sup\{|f(x) - g(x)| : x \in M\}$.

To prove (9.1), we use the argument given in [2, Lemma 3.1]: Pick $\delta = \|f - g\|_\infty$ and prove that $V(f)$ and $V(g)$ are δ-interleaved. Note that $V(f)[\delta] = V(f - \delta)$ and $V(g)[\delta] = V(g - \delta)$. By our choice of δ, we have $|f(x) - g(x)| \leq \delta$ for all $x \in M$. This implies that

$$g(x) - \delta \leq f(x) \leq g(x) + \delta \quad \text{and} \quad f(x) - \delta \leq g(x) \leq f(x) + \delta$$

for all $x \in M$. Therefore we also have

$$f(x) - 2\delta \leq g(x) - \delta \leq f(x) \quad \text{and} \quad g(x) - 2\delta \leq f(x) - \delta \leq g(x)$$

for all $x \in M$. Now the inclusions $L_t(f) \subseteq L_{t+\delta}(g)$ and $L_t(g) \subseteq L_{t+\delta}(f)$ for all t yield morphisms of persistence modules

$$F\colon V(f) \longrightarrow V(g)[\delta] \quad \text{and} \quad G\colon V(g) \longrightarrow V(f)[\delta],$$

and $F[\delta] \circ G$ is equal to the morphism induced by the inclusion $L_t(f) \subseteq L_{t+2\delta}(f)$, which is precisely the shift morphism $\sigma_{2\delta}$ for $V(f)$. The argument for $G[\delta] \circ F$ is the same, by symmetry. Hence we conclude that $V(f)$ and $V(g)$ are δ-interleaved by means of F and G, as needed.

As a corollary, if we denote by $D(f)$ the persistence diagram of $V(f)$, then

$$\boxed{W_\infty(D(f), D(g)) \leq \|f - g\|_\infty}$$

where W_∞ is the bottleneck distance.

10 Stability for Čech complexes

We apply (9.1) to the case when f and g are the distance functions on $\mathbb{R}^N$ to given point clouds X and Y. In this section we consider the Čech persistence module

$$\check{V}_t(X) = H_*(C_t(X); \mathbb{R}),$$

where $C_t(X)$ is the Čech complex of X. Note that, if $X = \{x_0, \ldots, x_n\}$, then

$$L_t(f) = \{p \in \mathbb{R}^n \mid d(p,X) \leq t\} = \bigcup_{i=0}^{n} \bar{B}_t(x_i) \simeq |C_{2t}(X)|.$$

Consequently,

$$V_t(f) = H_*(L_t(f);\mathbb{R}) \cong H_*(C_{2t}(X);\mathbb{R}) = \check{V}_{2t}(X),$$

or, equivalently, $\check{V}_t(X) = V_{t/2}(f) = V_t(2f)$. Similarly, $\check{V}_t(Y) = V_t(2g)$.

On the other hand,

$$\|f - g\|_\infty = \sup\{|d(p,X) - d(p,Y)| : p \in \mathbb{R}^N\} = d_H(X,Y),$$

where d_H denotes the Hausdorff distance. Thus we obtain that

$$d_{\text{int}}(\check{V}(X), \check{V}(Y)) = d_{\text{int}}(V(2f), V(2g)) \leq \|2f - 2g\|_\infty = 2\, d_H(X,Y).$$

However, there does not seem to be a similar inequality relating the Gromov–Hausdorff distance between X and Y (which is intrinsic) with the Čech persistence modules $\check{V}(X)$ and $\check{V}(Y)$. The difficulty with Čech complexes is that they are not solely determined by the table of distances between the points in the given point cloud, but they depend on the topology of the ambient space $\mathbb{R}^N$. For this reason, the interleaving distance between the persistence modules $\check{V}(X)$ and $\check{V}(Y)$ also depends on the ambient space.

Stability of persistence diagrams for the Hausdorff distance was first proved in [3].

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