INTRODUCTION TO BATALIN-VILKOVISKY ALGEBRAS

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1. Gerstenhaber Algebras and BV-algebras

In this section we will define Gerstenhaber and Batalin-Vilkovisky algebras. Gerstenhaber algebras mix a commutative operation and a "Lie bracket" by means of the Leibniz rule. Batalin-Vilkovisky algebras are Gerstenhaber algebras with a compatible square-zero operator Δ. Such algebraic structures appear in differential and complex geometry on polyvector fields. In particular, we will see that the algebra of polyvector fields tensored with the algebra of anti-holomorphic forms on a Calabi-Yau manifold carries the structue of a BV-algebra.

Definition 1.1. A *Gerstenhaber algebra* is a graded vector A space endowed with an associative graded-commutative product $\cdot : A \times A \to A$ of degree 0 and a Lie bracket of degree -1

$$[-,-] : A \times A \to A,$$

i.e. satisfies the following properties:

- $[a,b] = -(-1)^{|a-1||b-1|}[b,a]$,
- Jacobi identity: $[-,[-,-]] + [-,[-,-]^{(132)} + [-,[-,-]]^{(123)} = 0$ and
- Leibniz rule: $[- \cdot -, -] = ([-,-] \cdot -)^{(23)} + (- \cdot [-,-])$.

Gerstenhaber algebras differ from Poisson algebras in that the bracket is graded of degree -1.

Example 1.2. The space of polyvector fields of a differential manifold together with the Schouten–Nijenhuis bracket, obtained from the Lie bracket under the Leibniz rule.

Definition 1.3. A *Batalin-Vilkovisky algebra* (BV-algebra for short) is a graded vector space A, equipped with an associative graded-commutative product of degree 0,

$$\cdot : A \times A \to A$$

and an unary operation Δ of degree -1 that satisfies $\Delta \circ \Delta = 0$. These two operations satisfy:

$$\Delta(- \cdot - \cdot -) - (\Delta(- \cdot -) \cdot -)^{id+(123)+(321)} + (\Delta(-) \cdot - \cdot -)^{id+(123)+(321)} = 0.$$

Applied to elements $a,b,c \in A$ this gives:

$$\begin{aligned}&\Delta(abc) - \Delta(ab)c - (-1)^{|b|(|a|+|c|)}\Delta(ca)b - (-1)^{|a|(|b|+|c|)}\Delta(bc)a + \\ &\Delta(a)bc + (-1)^{|b|(|a|+|c|)}\Delta(c)ab + (-1)^{|a|(|b|+|c|)}\Delta(b)ca = 0.\end{aligned}$$

With such a structure one can define the following binary operation:

$$\langle -,- \rangle := \Delta(- \cdot -) - (- \cdot \Delta(-)) - (\Delta(-) \cdot -).$$

This bracket may be interpreted as an obstruction to Δ being a derivation for the commutative product.

Proposition 1.4. *The bracket $\langle -,- \rangle : A \times A \to A$ satisfies the following:*

(1) it is graded antisymmetric of degree -1,
(2) it satisfies the Jacobi identity,
(3) it satisfies the Leibniz rule with the product.

From the above proposition it follows that the couple of operations $(\cdot, \langle -,- \rangle)$ define a Gersternhaber algebra structure on A.

Proposition 1.5 ([Get94]). *The data of a BV-algebra structure $(A, \cdot, \Delta)$ is equivalent to that of a Gerstenhaber algebra structure $(A, \cdot, [-,-])$ endowed with a square-zero degree -1 unary operator Δ, such that*

$$[-,-] = \Delta(- \cdot -) - (- \cdot \Delta(-)) - (\Delta(-) \cdot -).$$

Definition 1.6. A *differential BV-algebra* is a BV-algebra $(A, \cdot, \Delta)$ together with a differential d satisfying

$$d \circ \Delta + \Delta \circ d = 0.$$

2. BV-algebra structure on the tensor bundle

In the previous talks we have seen that there is a differential graded Lie algebra structure on $\bigoplus A^{0,q}(X, T_X)$, in this section we will define the tructure of a Batalin-Vilkovisky algebra on $\bigoplus A^{0,q}(X, \Lambda^p T_X)$. The major part of this section is based on [Huy05].

Let X be a complex manifold. One defines,

$$A_X := \bigoplus A^{0,q}(X, \Lambda^p T_X).$$

Which is precisely the space of sections of the tensor bundle: $\Gamma(X, \Lambda^q \bar{T}_X^* \otimes \Lambda^p T_X)$. The exterior product determines the structure of a graded-commutative algebra on A_X.

Recall that if we have a holomorphic vector bundle $\pi : E \to X$, then $\bar{\partial}$ defines a differential on $A(X, E) := \Gamma(X, \Lambda^{p,q} X \otimes E)$. Which in local coordinates gives $\bar{\partial_E}(\Sigma \alpha_i \otimes s_i) = \Sigma \bar{\partial}(\alpha_i) \otimes s_i)$.

Thus, since the exterior powers $\Lambda^p T_X$ are holomorphic bundles, $\bar{\partial}$ endows A_X with the structure of a differential graded algebra.

On A_X we can define a Gerstenhaber bracket

$$[-,-] : A^{0,q}(\Lambda^p T_X) \times A^{0,q'}(\Lambda^{p'} T_X) \to A^{0,q+q'}(\Lambda^{p+p'-1} T_X),$$

which in local coordinates it is given by

$$\left[\alpha_{IJ} d\bar{z}_I \otimes \frac{\partial}{\partial z_J}, \beta_{KL} d\bar{z}_K \otimes \frac{\partial}{\partial z_L}\right] = d\bar{z}_I \wedge d\bar{z}_K \otimes \left[\alpha_{IJ} \frac{\partial}{\partial z_J}, \beta_{KL} \frac{\partial}{\partial z_L}\right]$$

with

$$[v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_{p'}] = \sum (-1)^{i+j} [v_i, w_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_{p'}$$

and for $p' = 0$,

$$[v_1 \wedge \cdots \wedge v_p, f] = \sum (-1)^{p-i} v_i(f) v_1 \wedge ... \wedge \hat{v}_i \wedge \cdots \wedge v_p.$$

This is a generalization of the Lie bracket introduced in the previous talks. The sign convention yields,

$$[\alpha, \beta] = -(-1)^{(p+q-1)(p'+q'-1)} [\beta, \alpha].$$

Lemma 2.1. *The bracket $[-,-] : A_X \times A_X \to A_X$ satisfies the Leibniz rule with the wedge product and the Jacobi identity.*

Thus, $(A_X, \wedge, [-,-])$ is a Gerstenhaber algebra.

Remark 1. The Gerstenhaber bracket gives sA_X (the suspension of A_X) a Lie superalgebra structure.

For any arbitrary manifold X the differential algebra $(A_X, \bar{\partial})$ has no reason to carry the additional structure of a BV-algebra. This is only ensured for Calabi-Yau manifolds.

Let X be a Calabi-Yau manifold of dimension n, this is a compact Kahler manifold of dimension n with a nowhere vanishing holomorphic n-form Ω.

This homolorphic form of degree n can be used to define a natural isomorphism

$$\eta : \Lambda^p T_X \cong \Omega_X^{n-p}.$$

If Ω is locally written in the form $f dz_1 \wedge ... \wedge dz_n$, then

$$\eta\left(\frac{\partial}{\partial z_{i_1}} \wedge ... \wedge \frac{\partial}{\partial z_{i_p}}\right) = (-1)^{(\Sigma i_j)-p} f dz_1 \wedge ... \wedge d\hat{z}_{i_1} \wedge ... \wedge d\hat{z}_{i_p} \wedge ... \wedge dz_n,$$

for $i_1 < ... < i_p$. Moreover, the isomorphism η induces canonical isomorphisms

$$\eta : A^{0,q}(\Lambda^p(T_X)) \cong A^{n-p,q}(X).$$

As in the previous talk we can define the operator $\Delta : A^{0,q}(\Lambda^p T_X) \to A^{0,q}(\Lambda^{p-1}T_X)$ as

$$\Delta : A^{0,q}(\Lambda^p T_X) \xrightarrow{\eta} A^{n-p,q}(X) \xrightarrow{\partial} A^{n-p+1,q}(X) \xrightarrow{\eta^{-1}} A^{0,q}(\Lambda^{p-1}T_X).$$

Lemma 2.2. *The operator Δ anti-commutes with $\bar{\partial}$ and it satisfies $\Delta^2 = 0$.*

Lemma 2.3 (Generalized Tian-Todorov lemma)**.** *For any $\alpha, \beta \in A_X$,*

$$[\alpha, \beta] = \Delta(\alpha\beta) - \Delta(\alpha)\beta - \alpha\Delta(\beta).$$

Therefore, by proposition 1.5 and lemma 2.2, $(A_X, \wedge, \Delta, \bar{\partial})$ define a differential BV-algebra structure on A_X.

3. The Gerstenhaber and BV operads

In order to connect this talk with the next one, we will briefly introduce the notion of operads and the construction of the Gerstenhaber and BV operads. For this section we will follow [LV12].

Let $Vect$ be category of vector spaces over the field $\mathbb{K}$ of characteristic 0.

Definition 3.1. A *collection* in $Vect$ is a family

$$M = (M(0), M(1), ..., M(n), ...),$$

of $\mathbb{K}$-vector spaces where each $M(n)$ is equipped with an action of the symmetric group Σ_n. For $\mu \in M(n)$, the integer n is called the *arity* of n.

Definition 3.2. Let M and N be two collections. The composition of collections is defined to be the collection

$$M \circ N := \bigoplus_{k\geq 0} M(k) \otimes_{\Sigma_k} N^{\otimes k}.$$

Definition 3.3. An *operad* is a collection $\mathcal{P} = \{P_n\}_{n\geq 0}$ together with linear maps

$$\gamma : P(k) \otimes P(i_1) \otimes ... \otimes P(i_k) \to P(n),$$

for all positive integers $k, i_1, ..., i_k$ where $n = i_1 + ... + i_k$, such that:

(a) Equivariance conditions:

$$\gamma(p\sigma; q_1, ..., q_n) = \gamma(p; q_{\sigma^{-1}(1)}, ..., q_{\sigma^{-1}(n)})\sigma'$$

where σ' refers to the element of $\Sigma_{k_1+\cdots+k_n}$ that acts on the set $\{1, 2, \ldots, k_1 + \cdots + k_n\}$ by breaking it into n blocks, the first of size k_1, the second of size k_2, through the nth block of size k_n, and then permutes these n blocks by σ, keeping each block intact. And

$$\gamma(p; q_1\tau_1, ..., q_k\tau_k) = \gamma(p; q_1, ..., q_k)(\tau_1, ..., \tau_k).$$

(b) Associativity condition:

$$\gamma(p; \gamma(q_1; r_{1_1}, \cdots, r_{1_{i_1}}), \gamma(q_2; r_{2_1}, \cdots, r_{2_{i_2}}), \cdots, \gamma(q_k; r_{k_1}, \cdots, r_{k_{i_k}})) = \gamma((\gamma(p; q_1, \cdots, q_k); r_{1_1}, \cdots, r_{k_{i_k}}).$$

(c) Unitality condition: there is an element $id \in \mathcal{P}(1)$ such that $\gamma(id; \mu) = \mu$ and $\gamma(\mu; id, ..., id) = \mu$.

We should think of an element in $\mathcal{P}(k)$ as an operation which has k inputs and 1 output and the maps γ as compositions of operations.

Definition 3.4. Let $\mathcal{P}$ and $\mathcal{Q}$ two operads. A *morphism* of operads from $\mathcal{P}$ to $\mathcal{Q}$ is a family of linear Σ_n-equivariant maps $f_n : \mathcal{P} \to \mathcal{Q}$ which are compatible with the composition maps.

Example 3.5.
- The operad End_V: For any vector space V the endomorphism operad End_V is given by $End_V(n) := Hom(V^{\otimes n}, V)$, where the action right action of Σ_n on $End_V(n)$ is given by the permuting the components of $V^{\otimes n}$ and the composition map is given by composition of endomorphisms.

- The operad *Com* is the operad encoding commutative algebras: This operad is defined to be $Com(n) = \mathbb{K}$ with trivial action and the composition are multiplication of elements in the ground field.

Definition 3.6. A *$\mathcal{P}$-algebra structure* on a vector space A is a morphism of operads $\mathcal{P} \to End(A)$.

There is the notion of *graded operad*, which is an operad whose underlying collection is defined over the category of graded $\mathbb{K}$-vector spaces. By definition, a *graded $\mathcal{P}$-algebra* structure over a graded vector space A is a morphism of graded operads $\mathcal{P} \to End_A$.

If $\mathcal{P}$ is a graded operad, then we can define the notion of the *operadic suspension,* which has the following property:

A graded vector space A is equipped with a $\mathcal{P}$-algebra structure if and only if the suspended vector space sA is equipped with a $\mathcal{S}\mathcal{P}$-algebra structure.

From now on, we will only consider $\mathbb{Z}_2$-gradings.

The operad *Gerst*: the operad *Gerst* is the one that defines Gerstenhaber algebras.

Recall: a Gerstenhaber algebra is a graded vector space endowed with a graded-commutative product and a Lie bracket of degree -1, moreover the Lie bracket satisfies Leibniz relation with the product.

By remark 1, the Gerstenhaber bracket on the suspension of the algebra gives a *Lie*-algebra structure, we also have that sA is a Lie algebra if and only if $A = s^{-1}(sA)$ is a $\mathcal{S}^{-1}Lie$-algebra.

The underlying collection of the operad *Gerst* is isomorphic to $Com \circ \mathcal{S}^{-1}Lie$, where the structure of operad in $Com \circ \mathcal{S}^{-1}Lie$ is given by means of distributive laws.

The idea is that *Com* defines the commutative product and $\mathcal{S}^{-1}Lie$ defines the Lie bracket, then the Leibniz relation is used to construct a morphism of collections

$$\lambda : \mathcal{S}^{-1}Lie \circ Com \to Com \circ \mathcal{S}^{-1}Lie$$

by sending $[- \cdot -, -] \mapsto ([-,-] \cdot -)^{(23)} + (- \cdot [-,-])$ which is used to define a on operad structure on the collection $Com \circ \mathcal{S}^{-1}Lie$. See [LV12] section 8.6 for details.

The *BV* **operad**: This operad encodes Batalin-Vilkovisky algebras.

Recall: A *BV*-algebra structure is equivalent to that of a Gerstenhaber algebra together with an operator Δ that squares to zero and such that the Gerstenhaber bracket acts as an obstruction to Δ being a derivation.

Then, the underlying collection of the operad *BV* is defined in a similar way, $BV \cong Gerst \circ D \cong Com \circ \mathcal{S}^{-1}Lie \circ D$ where D is the operad encoding the operator Δ.

References

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