Hypercommutative algebras

David Martínez Carpena

May 27, 2022

In these notes, we will present an introduction to hypercommutative algebras. This structure was discovered by Dijkgraaf, Verlinde and Verlinde [DVV91], and can be found with different names and small variations, like Witten-Dijkgraaf-Verlinde-Verlinde algebras, formal Frobenius manifolds, genus zero reduction of Gromov-Witten theories or genus zero cohomological field theories. The hypercommutative algebra structure plays a crucial role in mathematical physics because it describes the algebraic structure of quantum cohomology of varieties.

1 Operad of moduli spaces

In this section, we want to present the topological and geometrical structures underlying the hypercommutative algebras. To that end, we will define moduli spaces, its compactification, and the operadic structure found in the compactified ones of genus zero.

The *moduli space* $\mathcal{M}_{g,n}$ is the space of isomorphism classes of complex smooth projective curves of genus g with n distinct marked points. We are mainly interested in the case of genus zero, where $\mathcal{M}_{0,n}$ is an $(n-3)$-dimensional smooth variety.

An explicit definition of the case $\mathcal{M}_{0,n}$ is easy to find. Observe that the only complex curve of genus zero is the Riemann sphere, which can be thought as the complex projective line $\mathbb{C}\mathcal{P}^1$, and its group of isomorphisms is $\mathrm{PGL}_2(\mathbb{C})$. Therefore, for every $n \geq 3$, $\mathcal{M}_{0,n}$ is equivalent to the configuration space of n distinct labeled punctures on the complex projective line $\mathbb{C}\mathcal{P}^1$, considered up to the natural action of $\mathrm{PGL}_2(\mathbb{C})$. In addition, this definition can be simplified by the following isomorphism:

$$\begin{aligned}\mathcal{M}_{0,n} &= \left\{(p_1,\dots,p_n) \in (\mathbb{C}\mathcal{P}^1)^n \mid p_i \neq p_j \text{ for } i \neq j\right\}/\mathrm{PGL}_2(\mathbb{C})\\ &\cong \left\{(z_1,\dots,z_{n-3}) \in (\mathbb{C}\setminus\{0,1\})^{n-3} \mid z_i \neq z_j \text{ for } i \neq j\right\}\end{aligned}$$

Unfortunately, $\mathcal{M}_{g,n}$ is not a complete variety, because singular curves can appear as the limit of smooth curves. Deligne, Mumford and Knudsen [DM69; Knu83] constructed a compactification of $\overline{\mathcal{M}}_{g,n}$, by enlarging the moduli problem to include certain singular curves.

A *stable curve* C with n marked points is a complex projective curve such that:

- The only singularities are double points.
- There is n distinct smooth marked points $p_1,\dots,p_n \in C$.
- There are no continuous automorphisms of C fixing the marked and double points.

Then, the *compactified moduli space* $\overline{\mathcal{M}}_{g,n}$ is the space of isomorphism classes of stable curves of genus g with n distinct marked points. In the case of genus zero, $\overline{\mathcal{M}}_{0,n}$ is actually a projective variety.

Example 1.1.
- $\overline{\mathcal{M}}_{0,3}$ is a point.
- $\mathcal{M}_{0,4} \cong \mathbb{C}\mathcal{P}^1 \setminus \{0,1,\infty\}$ and $\overline{\mathcal{M}}_{0,4} \cong \mathbb{C}\mathcal{P}^1$.
- $\overline{\mathcal{M}}_{0,n+1}$ is inductively constructed as a blow up of $\overline{\mathcal{M}}_{0,n} \times \mathbb{C}\mathcal{P}^1$.

The moduli spaces of genus zero can be arranged as a topological operad $\overline{\mathcal{M}}_{0,\bullet+1}$: define the space of n-ary operations as $\overline{\mathcal{M}}_{0,n+1}$ for any $n \geq 2$, considering the first marked point as the output, and the operadic composition $\circ_i : \overline{\mathcal{M}}_{0,n+1} \otimes \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,n+m}$ as the gluing of stable curves at marked points.

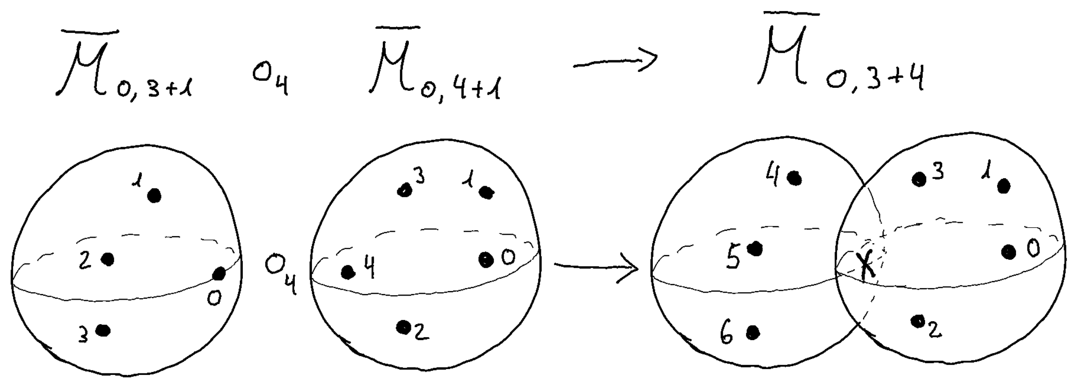

Figure 1: Example of composition of the operad $\overline{\mathcal{M}}_{0,\bullet+1}$.

2 Hypercommutative algebra

In this section we present hypercommutative algebras, their main properties, and its Koszul dual, the gravity algebras.

Definition 2.1. A *hypercommutative algebra* is a chain complex A with a sequence of totally symmetric n-ary operations $(x_1, \ldots, x_n) : A^{\otimes n} \to A$ of degree $2(n-2)$ for any $n \geq 2$, satisfying the following generalized associativity condition

$$\sum_{S_1 \sqcup S_2 = \{1,\ldots,k\}} (-1)^{\varepsilon}((a, b, x_{S_1}), c, x_{S_2}) = \sum_{S_1 \sqcup S_2 = \{1,\ldots,k\}} (-1)^{\varepsilon}(a, (b, c, x_{S_1}), x_{S_2}) \tag{1}$$

where $k \geq 0$, $a, b, c, x_1, \ldots, x_k \in A$, ε is the Koszul sign rule, and x_S denotes $x_{s_1}, \ldots, x_{s_m}$ for a finite set $S = \{s_1, \ldots, s_m\}$.

Example 2.2. Consider the generalized associativity conditions for small k values:

- If $k = 0$, the relation $(a, (b, c)) = ((a, b), c)$ is equivalent to the associativity of the binary operation. Therefore, the binary operation is graded commutative and associative.
- If $k = 1$, the relation is

$$(a, (b, c, d)) + (a, (b, c), d) = ((a, b), c, d) + (-1)^{|c||d|}((a, b, d), c).$$

One way of thinking about the operations in a hypercommutative algebra is to view them as the Taylor coefficients of a formal deformation of the commutative product (a, b). In any hypercommutative algebra there is a family of products $\{(a, b)_x\}_{x \in A}$ such that

$$(a, b)_x := \sum_{n \geq 0} \frac{1}{n!}(a, b, x, \ldots, x) \quad \text{for all } x \in A.$$

Then, the Equation 1 is equivalent to the associativity of the products $(a, b)_x$ for all $x \in A$. This characterization explains the relation between hypercommutative algebras and Frobenius manifolds.

Define the *HyperCom* operad as the one generated by an n-ary operation of degree $2(n-2)$ for any $n \geq 2$, and with the relations imposed by Equation 1. Then, by definition the algebras over the *HyperCom* operad are hypercommutative algebras.

Proposition 2.3. [KM94; Get95] *The operad $H_\bullet(\overline{\mathcal{M}}_{0,\bullet+1})$ formed by the homology of the compactified moduli space of genus 0 is isomorphic to the operad HyperCom encoding hypercommutative algebras.*

In the study of algebraic operads, Koszul duality (see [LV12, Chapter 7] for a detailed introduction) plays a central role. The main examples of Koszul duality are the associative operad, which is its own dual, and the operads *Comm* and *Lie*, which are dual to each other. As we have seen, a hypercommutative operad has a commutative binary product, but with extra structure. In the rest of this section we want to introduce the Koszul dual to the hypercommutative operad, which will be some kind of Lie operad with extra structure:

Definition 2.4. A *gravity algebra* is a chain complex A with a sequence of graded antisymmetric n-ary operations $[x_1, \ldots, x_n] : A^{\otimes n} \to A$ of degree $2-n$ for any $n \geq 2$, satisfying the following relations

$$\sum_{1\leq i<j\leq k} (-1)^{\varepsilon}[[a_i, a_j], a_1, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_k, b_1, \ldots, b_l] = \begin{cases} [[a_1, \ldots, a_k], b_1, \ldots, b_l] & l > 0 \\ 0 & l = 0 \end{cases}$$

where $k > 2$, $l \geq 0$, $a_1, \ldots, a_k, b_1, \ldots, b_l \in A$, and ε is the Koszul sign rule.

In this case, with $k = 3$ and $l = 0$, we obtain the Jacobi relation for $[a, b]$. As before, we can define the gravity operad *Grav* as the one generated by those operations and relations.

Observe that the moduli spaces $\mathcal{M}_{0,\bullet+1}$ have a similar structure to $\overline{\mathcal{M}}_{0,\bullet+1}$, but the gluing along one point of two smooth curves does not produce a smooth curve. However, this gluing endows an operadic structure to the suspension of the homology:

Proposition 2.5. [Get94] *The operad $sH_\bullet(\mathcal{M}_{0,\bullet+1})$ formed by the suspension of the homology of the moduli space of genus 0 is isomorphic to the operad Grav encoding gravity algebras, where*

$$sH_\bullet(\mathcal{M}_{0,\bullet+1}) := \begin{cases} \sum^{3-n}(\wedge^{max}\mathbb{C}^{\{1,\ldots,n\}}) \otimes H_\bullet(\mathcal{M}_{0,\bullet+1}) & n \geq 3, \\ 0 & n < 3. \end{cases}$$

Using Proposition 2.3, Proposition 2.5 and the mixed Hodge structure of the moduli spaces of curves, Getzler proved:

Theorem 2.6. [Get95] *The operads HyperCom and Grav are Koszul dual to each other.*

The existence of this Koszul duality is the main ingredient in the proof of the next section, and enables the relation of hypercommutative operads with BV operads.

3 Relation between *HyperCom* and *BV* operads

In this final section, we want to relate hypercommutative algebras with a homotopy quotient of BV-algebras by the BV-operator. This problem has a topological origin, because the two operads are homologies of some topological operads: the moduli spaces for hypercommutative algebras, and in the case of BV-operads, a homotopy quotient of the framed discs operad by the circle action. Because the two operads are formal [Gui+05], we can restrict ourselves to the study of the homologies.

First, we need to define what we mean for a homotopy quotient. Consider the category of topological spaces **Top** and the category of functors $\mathbf{Top}^{S^1}$ (category of spaces with an S^1-action), with S^1 considered as a category of one object and a morphism for each element of S^1. There exists a functor $Triv^{S^1} : \mathbf{Top} \to \mathbf{Top}^{S^1}$ defined by sending a topological space X to itself with the trivial action of S^1. Using Left Kan extension, we obtain an adjoint functor $\bullet/S^1$ to $Triv^{S^1}$ as a certain colimit. This adjoint coincides with the quotient of a space X by its action of S^1 when the action is free.

Consider the usual model structure in **Top**, where the fibrant replacement is trivial because all objects are fibrant, and the equivariant model structure on $\mathbf{Top}^{S^1}$, with the cofibrant replacement $\bullet \times ES^1$. The previous adjunction lifts to one in the homotopy categories, defined as follows:

$$(\bullet \times ES^1)/S^1 : \mathbf{Ho}(\mathbf{Top}^{S^1}) \rightleftarrows \mathbf{Ho}(\mathbf{Top}) : Triv^{S^1}$$

Then, the existence of this adjunction is equivalent to a natural isomorphism for all $X \in \mathbf{Top}^{S^1}$ and all $Y \in \mathbf{Top}$

$$\mathbf{Ho}(\mathbf{Top})((X \times ES^1)/S^1, Y) \cong \mathbf{Ho}(\mathbf{Top}^{S^1})(X, Triv^{S^1}(Y))$$

Therefore, we define a model for the homotopy quotient by S^1 as a functor $(X \times ES^1)/S^1$ such that is left adjoint to $Triv^{S^1}$ in the homotopy categories. Observe that we could also have taken any other cofibrant replacement in $\mathbf{Top}^{S^1}$ to obtain a different model of the homotopy quotient.

The same construction can be carried in the category of dg-operads **dgOp** instead of topological spaces. The cohomology ring of the circle is the Grassman algebra $\mathbb{K}[\Delta]$ with one odd generator of degree -1 such that $\Delta^2 = 0$. Then, $\mathbf{Top}^{S^1}$ its replaced by the category $\mathbf{dgOp}^{\Delta}$ of dg-operads with a chosen embedding of the Grassman algebra $\mathbb{K}[\Delta]$. Similar to the previous case, any dg-operad Q admits a trivial map $\mathbb{K}[\Delta] \to Q$ with $\Delta \to 0$, defining the functor $Triv^{\Delta} : \mathbf{dgOp} \to \mathbf{dgOp}^{\Delta}$.

Definition 3.1. The *homotopy quotient by* Δ is a functor $\bullet/\Delta : \mathbf{dgOp}^{\Delta} \to \mathbf{dgOp}$ such that for any pair $P \in \mathbf{dgOp}^{\Delta}$ and $Q \in \mathbf{dgOp}$, there exists a natural equivalence

$$\mathbf{Ho}(\mathbf{dgOp})((LP)/\Delta, Q) \cong \mathbf{Ho}(\mathbf{dgOp}^{\Delta})(P, Triv^{\Delta}(RY)),$$

where R (resp. L) is the fibrant (resp. cofibrant) replacement of **dgOp** (resp. $\mathbf{dgOp}^{\Delta}$).

Our goal is to define a map $\theta : (HyperCom, 0) \to (BV/\Delta, d)$. First, we want to define the image of the map (θ for each generator $m_n \in HyperCom(n)$, which by Proposition 2.3 represents the fundamental cycle $[\overline{\mathcal{M}}_{0,n+1}]$. The idea is to define the image $\theta(m_n)$ as a sum of all possible rooted trees of n-leaves, where the nodes with $n > 2$ inputs are replaced by iterations of the binary multiplication operation from BV/Δ. For the rest of the details (see [KMS13]) we need to choose a specific model of the homotopy quotient.

Theorem 3.2. [KMS13] *The map θ defined on generators extends to a quasi-isomorphism of operads $\theta : (HyperCom, 0) \to (BV/\Delta, d)$.*

Proof. The proof involves the construction of a zigzag of quasi-isomorphisms between $(HyperCom, 0)$ and $(BV/\Delta, d)$, using the gravity and the Gerstenhaber operads. Then, careful diagram chasing of that zigzag of quasi-isomorphisms allows us to proof that θ is in fact a quasi-isomorphism. □

From the zigzag of quasi-isomorphisms and the fact that *HyperCom* has trivial differential, we know that the cohomology of $(BV/\Delta, d)$ is isomorphic to $(HyperCom, 0)$. Connecting with the rest of the seminar, it can be shown that there is a Frobenius manifold structure on the cohomology of Calabi-Yau manifolds. By the previous result, we know that if the BV-algebra associated to a Calabi-Yau manifold has a trivialization of the BV-operator, then its cohomology will have a natural hypercommutative algebra structure, and therefore a Frobenious manifold structure.

References

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". In: *Publications mathématiques de l'IHÉS* 36.1 (Jan. 1969), pp. 75–109. DOI: 10.1007/bf02684599.

[DVV91] Robbert Dijkgraaf, Herman Verlinde, and Erik Verlinde. "Topological strings in d < 1". In: *Nuclear Physics B* 352.1 (Mar. 1991), pp. 59–86. DOI: 10.1016/0550-3213(91)90129-l.

[Get94] E. Getzler. "Batalin-Vilkovisky algebras and two-dimensional topological field theories". In: *Communications in Mathematical Physics* 159.2 (Jan. 1994), pp. 265–285. DOI: 10.1007/bf02102639.

[Get95] E. Getzler. "Operads and Moduli Spaces of Genus 0 Riemann Surfaces". In: *The Moduli Space of Curves*. Birkhäuser Boston, 1995, pp. 199–230. DOI: 10.1007/978-1-4612-4264-2_8.

[Gui+05] F. Guillen, V. Navarro, P. Pascual, and A. Roig. "Moduli Spaces and Formal Operads". In: *Duke Mathematical Journal* 129.2 (Aug. 2005). DOI: 10.1215/s0012-7094-05-12924-6.

[KM94] M. Kontsevich and Yu. Manin. "Gromov-Witten classes, quantum cohomology, and enumerative geometry". In: *Communications in Mathematical Physics* 164.3 (Aug. 1994), pp. 525–562. DOI: 10.1007/bf02101490.

[KMS13] A. Khoroshkin, N. Markarian, and S. Shadrin. "Hypercommutative Operad as a Homotopy Quotient of BV". In: *Communications in Mathematical Physics* 322.3 (June 2013), pp. 697–729. DOI: 10.1007/s00220-013-1737-7.

[Knu83] Finn F. Knudsen. "The projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$". In: *Mathematica Scandinavica* 52.2 (1983), pp. 161–199. DOI: 10.7146/MATH.SCAND.A-12001.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*. Springer Berlin Heidelberg, 2012. DOI: 10.1007/978-3-642-30362-3.