Algebraic and topological operads

We will start by briefly recalling the definition of operad and operadic algebra from the previous talk in a general context.

Let $(\mathscr{C}, \otimes, I, Hom)$ a closed symmetric monoidal category (e.g. sets, top. spaces, k-Vect, gr. k-Vect, chain complexes, ...).

Let Σ_n denote the symmetric group on n letters ($\Sigma_1 = \Sigma_0 = *$) and let $\mathscr{C}^{\Sigma_n}$ be the category of objects of $\mathscr{C}$ with an action of Σ_n.

$$\mathscr{C}^{\Sigma_n} = Fun(\Sigma_n, \mathscr{C}) \qquad \Sigma_n = \bullet \; \sigma \in \Sigma_n$$

We define the category of symmetric collections $Coll(\mathscr{C})$ as

$$Coll(\mathscr{C}) = \prod_{n \geq 0} \mathscr{C}^{\Sigma_n}$$

This is also called the category of S-modules by Getzler.

An object M in $\mathrm{Coll}(\mathcal{E})$ is of the form

$$M = \{ M(0), M(1), \dots, M(n), \dots \}$$
$$\circlearrowleft_{\Sigma_n}$$

The category $\mathrm{Coll}(\mathcal{E})$ admits a non-symmetric monoidal structure $\circ$

$$(M \circ N)(n) = \coprod_{k \geq 0} M(k) \otimes_{\Sigma_k} N^{\otimes k}(n),$$

where $N^{\otimes k}(n) = \coprod_{n_1 + \dots + n_k = n} \left(N(n_1) \otimes \dots \otimes N(n_k) \right) \otimes_{\Sigma_{n_1} \times \dots \times \Sigma_{n_k}} \Sigma_n$

The unit is $\mathbb{I} = \{ \emptyset, I, \emptyset, \dots \}$

An operad is a monoid in $(\mathrm{Coll}, \circ, \mathbb{I})$, that is, an object P together with a multiplication
$\mu : P \circ P \longrightarrow P$ and unit $\eta : \mathbb{I} \longrightarrow P$
such that

$$\begin{tikzcd}
P \circ P \circ P \arrow[r, "\mu \circ id"] \arrow[d, "id \circ \mu"'] \arrow[dr, phantom, "\circlearrowright"] & P \circ P \arrow[d, "\mu"] \\
P \circ P \arrow[r, "\mu"'] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I} \circ P \arrow[r, "\eta \circ id"] \arrow[dr, equal] & P \circ P \arrow[d] & P \circ \mathbb{I} \arrow[l, "id \circ \eta"'] \arrow[dl, equal] \\
& P &
\end{tikzcd}$$

19 May 2022 23:33

The multiplication $\mu : P \circ P \longrightarrow P$ translates into the composition product of the operad

$$\gamma : P(k) \otimes P(n_1) \otimes \cdots \otimes P(n_k) \longrightarrow P(n)$$

$$n = n_1 + \cdots + n_k$$

A morphism of operads is a morphism of monoids. There is a free-forgetful adjunction

$$F : \mathrm{Coll}(\mathcal{C}) \rightleftarrows \mathrm{Operads}(\mathcal{C}) : U$$

This allows us to build operads by "generators and relations"

For every object X in $\mathcal{C}$, let $\mathrm{End}(X)$ denote the symmetric collection

$$\mathrm{End}(X)(n) = \mathrm{Hom}(X \otimes \overset{(n)}{\cdots} \otimes X, X)$$

This is an operad called the endomorphism operad of X.

A P-algebra is an object $X \in \mathcal{C}$ together with a map of operads $P \longrightarrow \mathrm{End}(X)$

19 May 2022 23:33

Functors that preserve operads and algebras

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between symmetric monoidal categories. Which conditions do we need on F so that it preserves operads and algebras?

$$\begin{tikzcd}
FP(k) \otimes FP(n_1) \otimes \cdots \otimes FP(n_k) \arrow[rr, dashed, "?"] \arrow[d] & & FP(n) \\
F(P(k) \otimes P(n_1) \otimes \cdots \otimes P(n_k)) \arrow[urr, "F(\gamma)"'] & &
\end{tikzcd}$$

$F : \mathcal{C} \to \mathcal{D}$ is lax monoidal if there are natural maps

$$FX \otimes FY \longrightarrow F(X \otimes Y)$$

$$I_{\mathcal{D}} \longrightarrow F(I_{\mathcal{C}})$$

If F is lax monoidal, P is an operad in $\mathcal{C}$ and X is a P-algebra in $\mathcal{C}$, then:

(i) FP is an operad in $\mathcal{D}$

(ii) FX is an FP-algebra in $\mathcal{D}$.

19 May 2022 23:33

Important example: Let $\mathcal{C} = Top$ and let $\mathcal{D} = gr\ k\text{-}Vect$. Then taking homology is a lax monoidal functor

$$
\begin{tikzcd}
H_*(-;k): Top \arrow[rr] \arrow[d] & & gr\ k\text{-}Vect \\
C_*(-,k) \arrow[r] & Ch(k) \arrow[ur, "H_*(-,k)"'] &
\end{tikzcd}
$$

lax monoidal

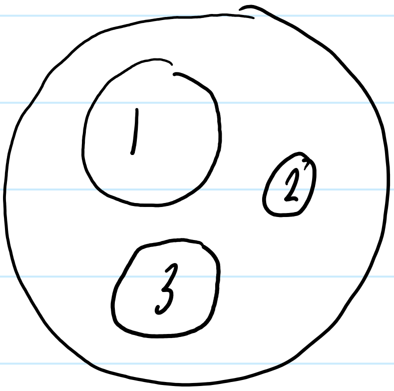
Thus, if P is an operad in Top and X is a P-algebra, then $H_*(X;k)$ is an $H_*(P;k)$-algebra.

Two examples of topological operads

① The little disks operad D_2

$D_2(n) = \{$ space of embeddings (composition of dilations and translations) of n disjoint 2-disks in a 2-disk $\}$

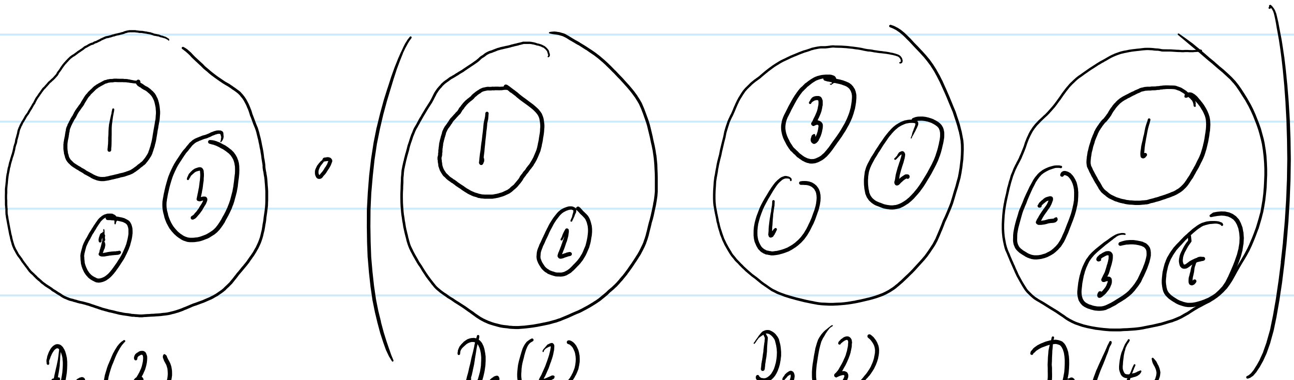
So, we consider embeddings $\coprod_n D^2 \longrightarrow D^2$ such that on each summand, the maps $D^2 \longrightarrow D^2$ is of the form $x \longmapsto \lambda x + c$, with $0 < \lambda < 1$.



$\in D_2(3)$

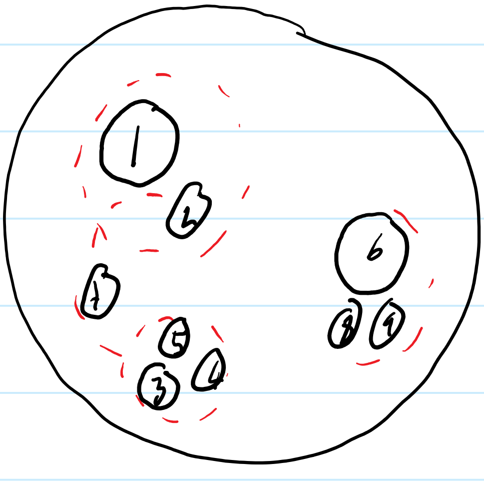
Σ_n acts on $D_2(n)$ by permuting the labels of the disks

Example of the composition product



$D_2(3) \circ \big(D_2(2) \quad D_2(3) \quad D_2(4) \big)$

$=$



$D_2(9)$

19 May 2022 23:33

- Theorem (Recognition principle, May, Boardman-Vogt)

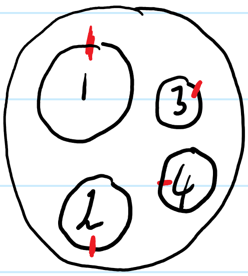
(i) Every 2-fold loop space is a D_2-algebra

(ii) If Y is a D_2-algebra and $\pi_0 Y$ is a group⊛ (e.g. X connected) then there exists a space X such that $Y \simeq \Omega^2 X$

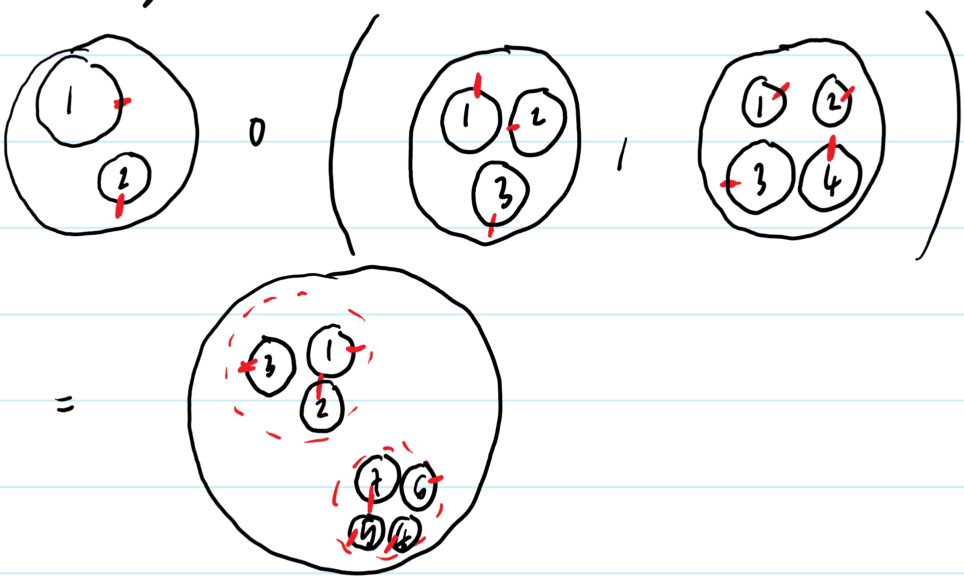
⊛ Every D_2-algebra has an underlying monoid structure on π_0

(2) The framed little disks operad fD_2

$fD_2(n)$ is the same as $D_2(n)$ but we also allow rotations

 $\in fD_2(4)$

Composition product



- **Theorem** (Recognition principle, Salvatore–Wahl)

(i) Every 2-folded loop space of an S^1-space is an fD_2-algebra

(ii) If Y is an fD_2-algebra and $\pi_0 Y$ is a group there exists an S^1-space X such that $Y \simeq \Omega^2 X$

Homology of operads

We are going to compute the homology of D_2 and fD_2.

Observe first that $D_2(n) \simeq Conf(n, \mathring{D}^2)$ the space of labelled configurations of n points in $\mathring{D}^2$ (send each disk to its center).

Taking $H_*(-, \mathbb{Q})$ we get

- **Theorem** (Cohen) $H_*(D_2, \mathbb{Q})$ is the operad Ger in $gr\,\mathbb{Q}\text{-Vect}$.

Recall that $Ger = Com \circ S^{-1} Lie$

$$\begin{tikzcd} Com\circ(S^{-1}Lie\circ Com)\circ S^{-1}Lie \arrow[r, dashed] \arrow[d, "\text{Leibniz}"'] & Com\circ S^{-1}Lie \\ Com\circ(Com\circ S^{-1}Lie)\circ S^{-1}Lie \arrow[ur] & \end{tikzcd}$$

19 May 2022 23:33

- Corollary: $H_*(\Omega^2 X;\mathbb{Q})$ is a Gerstenhaber algebra

- Theorem (Getzler). $H_*(fD_2;\mathbb{Q})$ is the operad BV.
  Recall that $BV = Gers \circ D$
  D operad encoding the operator Δ.