Deformations of Complex Manifolds

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1 Basic Definitions

In these notes, we will study deformations of the complex structure on a compact complex manifold and the DGLA associated to this deformation problem.

Fix some complex, compact manifold M of complex dimension m and let $B \subset \mathbb{C}^n$ be a small neighborhood of $0 \in \mathbb{C}^n$, for some n.

Definition 1. A **family of deformations of** M **over** B is a proper submersive holomorphic map $\pi : X \to B$ such that $M = \pi^{-1}(0)$.

Note that for each $t \in B$, the fibers $M_t = \pi^{-1}(t)$ are compact complex manifolds.

Definition 2. Two families $\pi : X \to B$ and $\pi' : X' \to B'$ are **equivalent** if there exists an open neighborhood $U \subset B \cap B'$ of $0 \in \mathbb{C}^n$ and a biholomorphism $\phi : X|_U \to X'|_U$ such that $\pi' \circ \phi = \pi$.

This definition of deformation becomes more intuitive after looking at the following theorem:

Theorem 3 (Ehresmann Fibration Theorem). *For such a map $\pi : X \to B$, there exists a diffemorphism $\psi : X \xrightarrow{\cong} M \times B$ such that $p_2 \circ \psi = \pi$, where $p_2 : M \times B \to B$ is the projection onto the second factor.*

Remark 4.

1. Note that this theorem implies that for every $t \in B$, the fiber M_t is diffeomorphic to M but not (necessarily) biholomorphic. One can thus think of a family of deformations as a family of copies of M with possibly different complex structures that vary holomorphically in $t \in B$.

2. Later, it will be useful to consider a refined version of Theorem 3. One can actually show that there exists a map $\phi : X \to M$ such that

   (a) $(\phi, \pi) : X \to M \times B$ is a diffeomorphism,

   (b) $\phi|_M = id_M$,

(c) The fibers of ϕ are holomorphic submanifolds of X transverse to M.

One calls the pair $(\phi, \pi) : X \to M \times B$ a **transversely holomorphic trivialization** of X.

Example 5 (Complex Torus)**.** Define the complex torus as $T = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, so it is the quotient of $\mathbb{C}$ by the lattice generated by 1 and i. One could instead consider the lattice generated by 1 and some other $\tau \in \mathbb{H}$ in the upper half plane. Such a choice will lead to a torus (as a smooth manifold) but with possibly a different complex structure. Let us define then the family $\pi : X \to \mathbb{H}$, where

$$X = \frac{\mathbb{C} \times \mathbb{H}}{(\omega, \tau) \sim (\omega + 1, \tau) \sim (\omega + \tau, \tau)}$$

and π is the projection onto the second factor. There exists a biholomorphism $\mathbb{H} \cong \mathbb{D}$ between the upper half plane and the unit disk that sends i to 0. One can then show that the composition of this biholomorphism with π satisfies the properties in Definition 1 and T is the fiber over 0. Thus, it forms a family of deformations of T. One can also prove that

$$T_\tau \cong_{\text{biholo.}} T_{\tau'} \qquad \text{iff} \qquad \tau = \frac{a + b\tau'}{c + d\tau'}, \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{Z}).$$

Hence, the torus has only on smooth structure but many complex ones.

2 Representing Deformations by a DGLA

Fix some family $\pi : X \to B$ and choose a transversely holomorphic trivialization $X \xrightarrow{\psi = (\phi, \pi)} M \times B$. Being X a complex manifold, it has associated an integrable almost complex structure J_X. Using $\psi = (\phi, \pi)$, one can then pushforward J_X to an integrable almost complex structure J' on $M \times B$,

$$J' = \psi_* J_X \psi_*^{-1}. \tag{1}$$

On the other hand, since both M and B are also complex manifolds, they too have associated integrable almost complex structures J_M and J_B, respectively, and so one has a product complex structure $J = J_M \times J_B$ on $M \times B$ (this J corresponds to the 'trivial' deformation, where one does not change the complex structure on M). We have thus two almost complex structures on $M \times B$ and we wish to compare them. Let us first see how to compare two almost complex structures on a general compact manifold and then apply the method to $M \times B$.

How to Compare Two Complex Structures: Let N be some compact smooth family and J and J' two almost complex structures on N. Recall that J induces a splitting of the complexified tangent bundle:

$$TN \otimes \mathbb{C} = T_{10} \oplus T_{01}, \qquad T_{10} = \overline{T_{01}}.$$

In fact, since T_{10} and T_{01} are the eigenspaces of J and $T_{10} = \overline{T_{01}}$, one has that J is also determined by T_{01}. Likewise, J' induces a splitting

$$TN \otimes \mathbb{C} = T'_{01} \oplus T'_{10}.$$

Now, if one assumes that T'_{01} is transverse to T_{10}, then the projection $\pi_{01} : TN \otimes \mathbb{C} \to T_{01}$ restricts to an isomorphism $T'_{01} \cong T_{01}$. One can then form the homomorphism

$$\underbrace{T_{01} \xrightarrow{(\pi_{01}|_{T'_{01}})^{-1}} T'_{01} \xrightarrow{\pi_{10}} T_{10}}_{\xi}.$$

Observe that $T'_{01} = \{v + \xi v | v \in T_{01}\}$ so, after fixing J, giving J' is equivalent to giving T'_{01} which is equivalent to giving ξ. Lastly, there is an isomorphism of vector bundles:

$$Hom(T_{01}, T_{10}) \cong (T_{01})^* \otimes T_{10} = A^{01} \otimes T_{10} = A^{01}(T_{10}),$$

and we will be looking at ξ as a section of $A^{01}(T_{10})$.

Let us now apply this to the complex structures J and J' on $M \times B$. Since, M_0 (the fiber of π over 0) is equal to M, J' restricted to $M \times 0$ is equal to J restricted to $M \times 0$. Thus, for $t = 0$, $T'_{01} = T_{01}$ and, using the fact that M is compact, one can shrink B so that over this shrunken neighborhood, J and J' are close enough and one can indeed apply the method just described. One thus gets a section

$$(\xi(t), \eta(t)) \in A^{01}(M \times B, T_{10}(M \times B)) \cong A^{01}(M \times B, T_{10}M) \oplus A^{01}(M \times B, T_{10}B).$$

Here, we use the notation $\xi(t)$ to emphasize the dependence of this section on the values of $t \in B$. By definition of J' (see (1) and using the fact that (1) π is holomorphic and submersive and (2) the fibers of ϕ are holomorphic submanifolds of X which are mapped isomorphically to B by π, one gets

$$T'_{01}(M \times B) = (\phi, \pi)_* T_{01}X = \phi_* T_{01}X \oplus T_{01}B.$$

It follows that $\eta(t) \equiv 0$ and $\xi(t)$ is zero on tangent vectors in $T_{01}B$. Moreover, $\xi(0) = 0$ because over $M \times 0$, J is equal to J'.

Remark 6. One can also view $\xi(t)$ as a smooth section of the pullback bundle of $A^{01}(M, T_{10}) \to M$ by the projection $p_1 : M \times B \to M$:

$$\begin{array}{ccc} A^{01}(M \times B, T_{10}) & \longrightarrow & A^{01}(M, T^{10}) \\ \xi(t) \uparrow\downarrow & & \downarrow \\ M \times B & \xrightarrow{p_1} & M \end{array}$$

such that $\xi(0) = 0$.

One should think of $\xi(t)$ as giving the difference, for each t, between the complex structure on $M_t \cong_{\text{diffeo}} M$ coming from J' and the initial complex structure on M.

From now one we will denote both $A^{01}(M \times B, T_{10})$ and $A^{01}(M, T_{10})$ by $A^{01}(T_{10})$ and hope that won't cause confusion.

3 DGLA of Forms and the Maurer-Cartan Equation

We have seen that for $B \subset \mathbb{C}^n$ small enough, giving an almost complex structure on $M \times B$ is equivalent to giving a smooth section $\xi(t)$ of $A^{01}(T_{10})$. There is however no reason for the almost complex structure corresponding to $\xi(t)$ to be integrable. It turns out that the sections that determine integrable complex structures are the solutions of the Maurer-Cartan equation.

Proposition 7. Consider the operations $[-,-] : A^{01}(T_{10}) \otimes A^{01}(T_{10}) \to A^{02}(T_{10})$ and $\overline{\partial} : A^{01}(T_{10}) \to A^{02}(T_{10})$ defined in local coordinates as

$$\left[\sum_{i,u} f_{iu} d\overline{z}_i \otimes \frac{\partial}{\partial z^u}, \sum_{j,v} g_{jv} d\overline{z}_j \otimes \frac{\partial}{\partial z^v}\right] = \sum d\overline{z}_i \wedge d\overline{z}_j \otimes \left[f_{iu}\frac{\partial}{\partial z^u}, g_{jv}\frac{\partial}{\partial z^v}\right],$$
$$\overline{\partial}\left(\sum_{i,u} f_{iu} d\overline{z}_i \otimes \frac{\partial}{\partial z^u}\right) = \sum \frac{\partial f_{iu}}{\partial \overline{z}^j} d\overline{z}^j \wedge d\overline{z}^i.$$

These are independent of the chosen coordinates and induce a DGLA structure on $\bigoplus_{i\geq 0} A^{0i}(T_{10})$.

Proposition 8. The section $\xi(t) \in A^{01}(T_{10})$ corresponds to an integrable almost complex structure over $M \times B$ iff the following are satisfied

1. $\overline{\partial}\xi(t) + \frac{1}{2}[\xi(t), \xi(t)] = 0$ (Maurer-Cartan equation),
2. $\xi(t)$ depends holomorphically in t.

Proof. See Theorem 1 of [Sch] □

Remark 9. In the previous proposition, condition (1) ensures that J' (the almost complex structure corresponding to $\xi(t)$) restricts to an integrable complex structure on each fibre M_t and condition (2) ensures that the complex structure varies holomorphically with t. We will be more concerned with condition (1).

Now we shall look for solutions of the Maurer Cartan equation:

$$\overline{\partial}\xi(t) + \frac{1}{2}[\xi(t), \xi(t)] = 0. \tag{2}$$

Let us assume that $B \subset \mathbb{C}$ and look first for solutions of the form $\xi(t) = \sum_{i\geq 1} \xi_i t^i$, with $\xi_i \in A^{01}(M, T_{10})$. We will focus only on finding formal solutions and not worry about convergence issues. Plugging $\xi(t) = \sum_{i \geq 1} \xi_i t^i$ in equation (2), one gets one equation for each power of t:

$$
\begin{aligned}
&\overline{\partial}\xi_1 = 0 \\
&\overline{\partial}\xi_2 + \frac{1}{2}[\xi_1, \xi_1] = 0 \\
&\overline{\partial}\xi_3 + [\xi_1, \xi_2] = 0 \\
&\vdots
\end{aligned}
\tag{3}
$$

The first equations tells us that, corresponding to a solution $\xi(t)$, there is a well defined class $[\xi_1] \in H^{01}_{Dol}(M, T_{10})$.

Definition 10. The cohomology class $[\xi_1] \in H^{01}_{Dol}(M, T_{10})$ is called the **Kodaira-Spencer class** of $\xi(t)$.

Remark 11.

1. One can define the Kodaira-Spencer class more generally for solutions of equation (2) which are not necessarily given by a power series. For the definition see page 6 of [Sch].
2. Recall that, given a family of $\pi : X \to B$, the definition of the corresponding $\xi(t)$ was contingent upon a choice of transversely holomorphic trivialization $X \xrightarrow{\psi} M \times B$. Given another choice $X \xrightarrow{\psi'} M \times B$, one has a smooth automorphism of $M \times B$:

$$
\begin{array}{ccc}
M \times B & \xrightarrow{(x,t) \mapsto (f_t(x), t)} & M \times B \\
& {\nwarrow}^{\psi} \quad {\nearrow}^{\psi'} & \\
& X &
\end{array}
$$

   where f_t is a family of smooth automorphisms of M such that $f_0 = id_M$. For such a family, one can show (see page 78 of [GHJ03]) that

$$
\xi_1' - \xi_1 = \overline{\partial}\left(\frac{d}{dt} f_t\right)_{10}.
$$

   Therefore, $[\xi_1'] = [\xi_1] \in H^{01}_{Dol}(M, T_{10})$.
   This implies in particular that equivalent deformations have the same Kodaira-Spencer class.

Now, given some class $\alpha \in H^{01}_{Dol}(M, T_{10})$, one can ask if there exists a solution $\xi(t)$ such that $[\xi_1] = \alpha$. Notice that this is not true in general. Looking at the second equation in (3), one sees that α must satisfy $[\alpha, \alpha] = 0$. And if α does satisfy that equation, one can choose an element ξ_2 such that $\overline{\partial}(2\xi_2) = [\xi_1, \xi_1]$ and then the next equation implies that $[\xi_1, \xi_2]$ must be exact. At each step, one must choose an element ξ_n and there might be a set of choices that leads to a solution and another that doesn't. However, observe that all these obstructions lie in $H^{02}_{Dol}(M, T_{10})$.

Theorem 12. *Let M be a compact complex manifold such that $H^{02}_{Dol}(M, TM_{10}) = 0$. Then, there exists an open neighborhood $B \subset \mathbb{C}^n$ such that for all $\alpha \in H^{01}(M, TM_{10})$, there exists a family of deformations $\pi : X \to B$ with $[\xi_1] = \alpha$*

Proof. See Theorem 2 of [Sch] □

Theorem 13. *If M is a Calabi-Yau manifold, then for every $\alpha \in H^{01}_{Dol}(M, TM_{10})$, there exists a solution $\xi(t)$ of the Maurer-Cartan equation such that $[\xi_1] = \alpha$.*

Proof. See Proposition 6.1.11 of [Huy05]. □

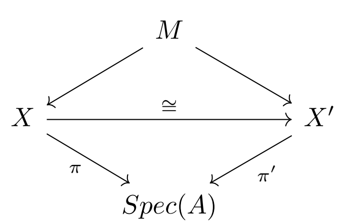
4 Infinitesimal Deformations

We have seen that in general $H^{01}(M, TM_{10})$ doesn't parameterize all deformations of M. But, it parameterizes all infinitesimal first order deformations. The definitions and results of this section (and the details necessary to understand them) can be found in chapters I and II of [Iac07].

Definition 14. Let M be a compact, complex manifold. An **infinitesimal deformation of M over $Spec(A)$** is a proper, flat, holomorphic map $\pi : X \to Spec(A)$ of complex spaces, with A a local Artinian $\mathbb{C}$-algebra and such that $M \cong X \times_{Spec(A)} Spec(\mathbb{C})$.

Definition 15. An **infinitesimal first order deformation of M** is an infinitesimal deformation of M over $Spec(\mathbb{C}[t]/(t^2))$.

Definition 16. Two infinitesimal deformations of M, $\pi : X \to Spec(A)$ and $\pi' : X' \to Spec(A)$ are said to be **isomorphic** if there exists an isomorphism of complex spaces $X \to X'$ making the following diagram commute:



Proposition 17. The functor $Def : \mathbf{Art} \to \mathbf{Set}$ given by

$$Def(A) = \{ \text{ infinitesimal deformations } \pi : X \to Spec(A) \text{ of } M\}/\cong$$

is a deformation functor.

Theorem 18. *The functor Def is naturally isomorphic to the deformation functor Def_L associated to the DGLA $L = (\bigoplus_{i\geq 0} A^{0i}(T_{10}), [-,-], \overline{\partial})$.*
In particular, for $A = \mathbb{C}[t]/(t^2)$, one has

$$\frac{\{\text{ inf. defor. } \pi : X \to Spec(\mathbb{C}[t]/(t^2)) \text{ of } M\}}{\cong} \cong \frac{\{\xi \in A^{01}(M, T^{10}) \otimes (t) | \overline{\partial}\xi + \frac{1}{2}[\xi,\xi] = 0\}}{\sim}$$
$$= H^{01}_{Dol}(M, T_{10}).$$

Proof. See Theorem II.7.3 of [Iac07]. □

References

[GHJ03] M. Gross, D. Huybrechts, and D. Joyce. *Calabi-Yau Manifolds and Related Geometries*. Universitext. Springer-Verlag, 2003.

[Huy05] D. Huybrechts. *Complex geometry*. Universitext. Springer-Verlag, Berlin, 2005.

[Iac07] Donatella Iacono. "Differential Graded Lie Algebras and Deformations of Holomorphic Maps". PhD thesis. Universita degli Studi di Roma, 2007.

[Sch] C. Schnell. *Deformations of Complex Structures*.