DEFORMATION THEORY VIA DGLA'S

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Let $\mathbb{K}$ be a commutative field of characteristic zero.

References for this talk can be found in [Man05] and [Man06].

1. Differential graded Lie algebras

Definition 1. A **differential graded Lie algebra** (DGLA for short) is a graded vector space $L = \bigoplus_{n\in\mathbb{Z}} L^n$ together with a differential d of degree $+1$ and a bilinear map $[-,-] : L^{\otimes 2} \to L$ satisfying

(i) Graded anti-symmetry: $[x, y] = -(-1)^{|x||y|}[y, x]$ for $x, y \in L$ of homogeneous degree.
(ii) Graded Jacobi identity: $[x,[y,z]] = [[x,y],z]+(-1)^{|x||y|}[y,[x,z]]$ for $x,y,z \in L$ of homogeneous degree.
(iii) Graded Liebniz rule: $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$ for $x,y \in L$ of homogeneous degree.

A morphism of DGLAs from L to M is a degree-preserving linear map $f : L \to M$ such that $fd_L = d_M f$ and $f[-,-] = [f,f]$.

Example 2. Given a differential graded associative algebra $A = \bigoplus_{n\in\mathbb{Z}} A^n$, the commutator $[a,b] := ab - (-1)^{|a||b|}ba$ gives the structure of DGLA on A.

Example 3. Given a cochain complex $V = \bigoplus_{n\in\mathbb{Z}} V^n$, the space of endomorphisms $\mathrm{End}(V) := \bigoplus_{k\in\mathbb{Z}} \mathrm{End}^k(V)$, where

$$\mathrm{End}^k(V) := \{f : V \to V : f(V^i) \subseteq f(V^{i+k})\},$$

can be given the structure of a differential graded associative algebra by letting

$$fg = f\cdot g := f\circ g, \quad \text{and} \quad d(f) := d\circ f - (-1)^{|f|} f\circ d.$$

The commutator defined in the previous example gives the structure of a DGLA on $\mathrm{End}(V)$.

Example 4. Given a DGLA $L = \bigoplus_{n\in\mathbb{Z}} L^n$ and a commutative algebra A, $L\otimes A$ can be given the structure of DGLA as follows:

$$[x\otimes a, y\otimes b] := [x,y]\otimes ab, \quad \text{and} \quad d(x\otimes a) := dx\otimes a.$$

Definition 5. A map of DGLAs $f : L \to M$ is said to be a **quasi-isomorphism** if the induced map $H^*(f) : H^*(L) \to H^*(M)$ is an isomorphism. Two DGLAs L and M are **weakly equivalent** if there exists a string of quasi-isomorphisms $L \leftarrow \cdots \to M$. We say that a DGLA L is **formal** if it is weakly equivalent to $(H^*(L), 0)$.

2. Deformation of a cochain complex

We want to deform a cochain complex $(C = \{C^n\}_{n\in\mathbb{Z}}, \partial)$ by tensoring with the dual numbers $\mathbb{K}[t]/(t^2)$. We have to think of t as a small parameter, so small that it vanishes when it is squared.

If

$$\cdots \xrightarrow{\partial} C^{-1} \xrightarrow{\partial} C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \cdots$$

is our cochain complex, the deformed cochain complex will be $C \otimes \mathbb{K}[t]/(t^2)$, i.e.

$$\cdots \xrightarrow{\tilde\partial} C^{-1} \otimes \mathbb{K}[t]/(t^2) \xrightarrow{\tilde\partial} C^0 \otimes \mathbb{K}[t]/(t^2) \xrightarrow{\tilde\partial} C^1 \otimes \mathbb{K}[t]/(t^2) \xrightarrow{\tilde\partial} \cdots,$$

with $\tilde\partial = \partial + \xi$ a perturbed differential. The perturbation ξ has to be an element of $\mathrm{End}^1(C) \otimes (t)$ because we want it to vanish modulo (t). It clearly is of degree $+1$, and it squares to zero if and only if,

$$0 = \tilde\partial^2 = (\partial + \xi)^2 = \partial\xi + \xi\partial + \xi^2 = d\xi + \frac{1}{2}[\xi, \xi]$$

in the Lie algebra $\mathrm{End}(C) \otimes (t)$.

Therefore,

$$\tilde\partial = \partial + \xi \text{ is a differential} \quad \Longleftrightarrow \quad \xi \text{ satisfies the Maurer-Cartan equation.}$$

If we write $\xi = f \otimes t$ (without loss of generality), then $\frac{1}{2}[\xi, \xi] = \xi^2 = (f \otimes t)^2 = f^2 \otimes t^2 = 0$, so in fact ξ only needs to be of the form $f \otimes t$ with f a cycle (i.e. $df = 0$).

Now, suppose we want to deform C tensoring by $\mathbb{K}[t]/(t^3)$ instead of the dual numbers. Both the product and the differential are defined analogously, and every condition is satisfied except from $\tilde\partial^2 = 0$. In place of repeating the process all over again, we can try to lift a solution ξ obtained before via the projection

$$\mathrm{End}^1(C) \otimes (\mathbb{K}t \oplus \mathbb{K}t^2) \xrightarrow{\pi} \mathrm{End}^1(C) \otimes \mathbb{K}t.$$

Let $\tilde\partial = \partial + \xi$, with $\xi \in \mathrm{End}^1(C) \otimes \mathbb{K}t$, be a differential of $C \otimes \mathbb{K}[t]/(t^2)$. When does there exist some $\bar\xi \in \mathrm{End}^1(C) \otimes (\mathbb{K}t \oplus \mathbb{K}t^2)$ such that $\bar\partial := \partial + \bar\xi$ is a differential of $C \otimes \mathbb{K}[t]/(t^3)$, and $\pi(\bar\xi) = \xi$?

Without loss of generality, we can write $\xi = f \otimes t$. Then

$$\pi^{-1}(\xi) = \{f \otimes t + g \otimes at^2 \in \mathrm{End}^1(C) \otimes (\mathbb{K}t \oplus \mathbb{K}t^2) : a \in \mathbb{K},\ g \in \mathrm{End}^1(C)\}.$$

Suppose $\tilde\partial$ is a differential, i.e. $df = 0$. Then $\bar\partial$ is a differential iff $\bar\xi = f \otimes t + g \otimes at^2$ also satisfies the Maurer-Cartan equation:

$$\begin{aligned} 0 = d\bar\xi + \tfrac{1}{2}[\bar\xi, \bar\xi] &= df \otimes t + dg \otimes at^2 + \tfrac{1}{2}[f, f] \otimes t^2 + \tfrac{1}{2}\left([f, g] + [g, f] + [g, g]\right) \otimes t^3 p(t) \\ &= \tfrac{1}{2}[f, f] \otimes t^2 + dg \otimes at^2 \\ &= f^2 \otimes t^2 + d(ag) \otimes t^2, \end{aligned}$$

which is equivalent to $f^2 = d(-ag)$.

One could keep trying to lift the perturbation to $\mathrm{End}^1(C) \otimes \mathbb{K}[t]/(t^n)$ by an analogous procedure, or even $\mathrm{End}^1(C) \otimes \mathbb{K}[[t]]$.

3. Local Artinian algebras and deformation functors

In order to work with full generality, we will state the abstract theory in the language local Artinian algebras in place of $\mathbb{K}[t]/(t^n)$ or $\mathbb{K}[[t]]$.

Definition 6. An **Artinian ring** is a commutative ring that satisfies the descending chain condition of ideals: every descending chain of left or right ideals $I_0 \supseteq I_1 \supseteq \cdots$ stabilizes at some n. A **local Artinian $\mathbb{K}$-algebra** A is a finitely generated $\mathbb{K}$-algebra which is also an Artinian ring, which has a unique maximal ideal, $\mathfrak{m}_A$, and with a splitting $A = \mathfrak{m}_A \oplus \mathbb{K}$ as $\mathbb{K}$-vector spaces. We say that $\mathbb{K}$ is the **residue field** of A, and it satisfies $A/\mathfrak{m}_A \cong \mathbb{K}$.

The category of local Artinian $\mathbb{K}$-algebras with morphisms being the morphisms of $\mathbb{K}$-algebras will be denoted by **Art**.

Example 7. The following are local Artinian algebras: $\mathbb{K}[t]/(t^n)$ with $n > 1$, $\mathbb{K}[[t]]$ and the fields.

Definition 8. A **deformation functor** is a functor $F : \mathbf{Art} \to \mathbf{Set}$ such that, if $\eta : F(B \times_A C) \to F(B) \times_{F(A)} F(C)$, it satisfies:

(i) $F(\mathbb{K}) = *$.
(ii) If $B \to A$ is surjective, so is η.
(iii) If $A = \mathbb{K}$, η is an isomorphism.

3.1. The functor $\mathbf{MC}_L$.

Definition 9. The functor $\mathrm{MC}_L : \mathbf{Art} \to \mathbf{Set}$ is defined by

$$\mathrm{MC}_L(A) := \{x \in L^1 \otimes \mathfrak{m}_A : dx + \frac{1}{2}[x, x] = 0\}.$$

It can be proven it satisfies the definition of deformation functor.

Remark 10. $L \otimes \mathfrak{m}_A$ is a DGLA because $\mathfrak{m}_A$ is commutative (see Example 4). This is because we restricted to commutative local Artinian algebras, eventhough it is not needed. In the general setting, the defined operation $[-,-]$ is not a Lie bracket (it does not anticommute) but is still used.

Example 11. As a computational example, we compute the tangent space of MC_L. The tangent space is defined as $t_{\mathrm{MC}_L} := \mathrm{MC}_L(\mathbb{K}[t]/(t^2))$. An element $x \otimes at$ belongs to t_{MC_L} if and only if

$$0 = dx + [x \otimes at, x \otimes at] = dx + x^2 \otimes a^2 t^2 = dx \iff x \in Z^1(L),$$

so the tangent space is $t_{\mathrm{MC}_L} = Z^1(L) \otimes \mathbb{K}t$.

There are some solutions of the Maurer-Cartan equation that will be understood to be isomorphic. This gives an equivalence relation in $\mathrm{MC}_L(A)$ for every A, which can be stated in two equivalent forms:

- Two elements $x, y \in \mathrm{MC}_L(A)$ are equivalent if and only if there exists an element $a \in L^0 \otimes \mathfrak{m}_A$ such that

$$y - x = \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n+1)!}([a, x] - da).$$

- Let $\mathbb{K}[t, dt]$ be the commutative differential graded algebra generated by t (of degree 0) such that $d(t) = dt$. We can form the DGLA of paths $\Omega := L \otimes \mathbb{K}[t, dt]$, $\Omega^n = L^n[t] \oplus L^{n-1}[t]dt$, with differential
$$\delta : \begin{cases} at^n \mapsto d(a)t^n + (-1)^{|a|} ant^{n-1}dt \\ at^n dt \mapsto d(a)t^n dt, \end{cases}$$
extended linearly, and with Lie bracket
$$[a(t) + b(t)dt, p(t) + q(t)dt] := \\ = [a(t), p(t)] + [a(t), q(t)dt] + (-1)^{|p(t)|}[b(t), q(t)]dt,$$
where $a(t), b(t), p(t), q(t) \in L[t]$.

  Given $k \in \mathbb{K}$, there is the evaluation morphism $v_k : L \otimes \mathbb{K}[t, dt] \to \mathbb{K}$ that sends $t \mapsto k$ and $dt \mapsto 0$. This allows us to define an equivalence relation in $\mathrm{MC}_L(A)$, saying that x and y are equivalent if and only if there exists some $\omega \in \mathrm{MC}_\Omega(A)$ such that $v_0^*(\omega) = x$ and $v_1^*(\omega) = y$ (we denoted $v_k^* = \mathrm{MC}_L(v_k)$).

In any of these cases, we write $x \sim y$ if and only if x is equivalent to y.

Example 12. Let L be a DGLA. The defined equivalence relation on $\mathrm{MC}_L(\mathbb{K}[t]/(t^2))$ can be shown to be $a \sim b$ if and only if $b - a$ is a boundary, using what was deduced in Example 11.

3.2. The functor Def_L.

The equivalence relation $\sim$ allows us to define the quotient functor of MC_L:

Definition 13. Let L be a DGLA. We define the following deformation functor: $\mathrm{Def}_L : \mathbf{Art} \to \mathbf{Set}$, $\mathrm{Def}_L(A) := \mathrm{MC}_L(A)/\sim$.

4. Some deformation results

Proposition 14. *If L and M are weakly equivalent DGLAs then $Def_L \cong Def_M$.*

Some DGLAs allow for simplifications of the Maurer-Cartan equation.

Corollary 15. *Let L be a DGLA.*

(i) *If L is formal then $MC_L(A) \cong \{x \in H(L)^1 \otimes \mathfrak{m}_A : [x, x] = 0\}$.*
(ii) *If L is quasi-abelian (i.e. quasi-isomorphic to some DGLA L' with vanishing Lie bracket) then $MC_L(A) \cong \{x \in (L')^1 \otimes \mathfrak{m}_A : dx = 0\}$.*

Proposition 16. *If L is a formal DGLA, the projections*
$$Def_L(\mathbb{K}[t]/(t^3)) \xrightarrow{\phi} Def_L(\mathbb{K}[t]/(t^2)),$$
$$Def_L(\mathbb{K}[[t]]) \xrightarrow{\psi} Def_L(\mathbb{K}[t]/(t^2))$$
have the same image, which is $\{f \otimes t : [f, f] = 0\}$.

Proof. Because $\mathrm{Def}_L(A)$ is a quotient of $\mathrm{MC}_L(A)$ for every local Artinian algebra A, it suffices to prove the result for the functor MC_L instead of Def_L. Also, since L is formal the Maurer-Cartan equation becomes $[x, x] = 0$.

An element $f \otimes t + g \otimes t^2 \in L^1 \otimes (\mathbb{K}t \oplus \mathbb{K}t^2)$ satisfies the Maurer-Cartan equation iff

$$0 = [f \otimes t + g \otimes t^2, f \otimes t + g \otimes t^2] = [f, f] \otimes t^2,$$

which is equivalent to $[f, f] = 0$. Hence,

$$\mathrm{im}(\phi) = \{f \otimes t : [f, f] = 0\}.$$

An element $f \otimes t \in \mathrm{im}(\phi)$ is clearly in $\mathrm{im}(\psi)$, since it satisfies the Maurer-Cartan equation in $L^1 \otimes (\bigoplus_{n\geq 1} \mathbb{K}t^n)$.

It remains to show the inclusion $\mathrm{im}(\psi) \subseteq \mathrm{im}(\phi)$. An element $\sum_{k\geq 1} f_k \otimes t^k$ satisfies the Maurer-Cartan equation iff

$$0 = \left[\sum_{k\geq 1} f_k \otimes t^k, \sum_{k\geq 1} f_k \otimes t^k \right] = \sum_{k\geq 1} \left(\sum_{p=1}^{k-1} [f_p, f_{k-p}] \right) \otimes t^k,$$

which is equivalent to

$$\sum_{p=1}^{k-1} [f_p, f_{k-p}] = 0 \qquad \forall k \geq 2.$$

In particular, when $k = 2$ we recover the equation $[f_1, f_1] = 0$, so $\mathrm{im}(\psi) \subseteq \mathrm{im}(\phi)$.

□

This proposition gives us an easy way to know what solutions lift from $\mathrm{Def}_L(\mathbb{K}[t]/(t^2))$ to $\mathrm{Def}_L(\mathbb{K}[[t]])$. In particular, these solutions are exactly those that lift from $\mathrm{Def}_L(\mathbb{K}[t]/(t^2))$ to $\mathrm{Def}_L(\mathbb{K}[t]/(t^3))$, which are $\{f \otimes t : [f, f] = 0\}$.

References

[Man05] Marco Manetti, *Deformation theory via differential graded lie algebras.*

[Man06] ———, *Differential graded lie algebras and formal deformation theory.*