Symmetry-Breaking in two-player games via strategic substitutes and diagonal nonconcavity:
A synthesis*

Rabah Amir†, Filomena Garcia‡, Malgorzata Knauff§

November 2009

Abstract
This paper is an attempt to develop a unified approach to symmetry-breaking in strategic models arising in industrial organization by constructing two general classes of two-player symmetric games that always possess only asymmetric pure-strategy Nash equilibria. These classes of games are characterized in some abstract sense by two general properties: payoff non-concavities and some form of strategic substitutability. Our framework relies on easily verified assumptions on the primitives of the game, and relies on the theory of supermodular games. The underlying natural assumptions are satisfied in a number of two-stage models with an investment decision preceding product market competition. To illustrate the generality and wide scope for application of our approach, we

*This paper has benefitted from comments by Claude d’Aspremont, Jean-Francois Mertens, Stanley Reynolds, Jacques Thisse, and John Wooders. We would also like to thank two anonymous referees and in particular an associate editor of this Journal for precise, thoughtful and detailed reports on an earlier version that have significantly improved the exposition of this paper.
†Department of Economics, University of Arizona, Tucson, AZ 85721, USA.
‡ISEG/Technical University of Lisbon and UECE, Rua Miguel Lúpi, 20, 1249-078 Lisbon, Portugal.
§Warsaw School of Economics, Al. Niepodleglosci 162, 02-554 Warsaw, Poland.
present some existing models dealing with R&D, capacity expansion and information provision, which motivated this study.

**Keywords**: submodular games, endogenous heterogeneity, asymmetric Nash equilibrium, inter-firm heterogeneity, supermodular games.

**JEL Codes**: C72, C62, L11.
1 Introduction

Inter-agent heterogeneity is often a critical dimension of various economic and social phenomena. From a positive perspective, heterogeneity is a necessary postulate to account for the simple fact that in the real world, one seldom observes identical agents, be it individuals, firms, industries or countries. From a normative standpoint, differences across interacting agents often form a necessary condition for many key economic activities such as trade and risk-sharing.

Understanding the origins and evolution of diversity across economic agents or disparities in economic performance across economies is increasingly perceived as a central goal of research in different areas of economics and social sciences. Macroeconomists seek to explain the causes of booms and recessions. Development economists grapple with the forces behind strong and poor economic performances of different regions or countries. Labor economists attempt to get a handle on discriminatory treatment of groups of workers. Business strategists and industrial economists devote a lot of attention to the sources and sustainability of inter-firm heterogeneity within and across industries. Overall, much effort has been expanded with a view to explain "diversity across space, time and groups" in economic settings (Matsuyama, 2002).

In the physical sciences, the concept of symmetry-breaking has been invoked to address a wide variety of phenomena in a long and illustrious history going back to the nineteenth century (e.g. to Pierre Curie). Some insightful similarities in the use of this concept in the natural sciences and the social sciences are explored in Matsuyama (2005), who also provides an intuitive comparative perspective on the emergence of symmetry-breaking through the conjunction of antagonistic forces in economics and in physics.\(^1\)

In view of the diversity of economic research areas invoking symmetry break-

\(^1\)A survey of the literature on the concept of symmetry breaking in physics is in [39], also see Anderson (1972). From these readings, one can easily grasp that the dichotomy between symmetry and asymmetry is much richer in the natural sciences than it is in economics. The economics literature is more closely related to the more specific concept of "spontaneous symmetry breaking" covered in Section 4.2 of [39].
ing, several conceptual and methodological approaches have been developed. While often tailored to a specific area, each of these approaches is broad in explanatory scope and has wide potential applicability.

The conventional approach, based on coordination failures, postulates a game with strategic complementarities and multiple Pareto-ranked pure-strategy Nash equilibrium points. Diversity is then synonymous with making different equilibrium selections, with the high-performing entity picking the Pareto-dominant equilibrium and the low-performing entity failing to do so. This argument is thus generally predicated on the presence of two identical and non-interacting economies, each operating under a different equilibrium out of the same equilibrium set. It may also be invoked to explain diversity across time within the same economy, with booms and recessions corresponding to operation under the Pareto dominant and inferior equilibria respectively. See Cooper and John (1988), Murphy, Schleifer and Vishny (1989), and Cooper (1999).

The coordination failures approach has been criticized for failing to offer any compelling argument for the diversity in equilibrium selections in the two-economy model or for the regime switch in the one-economy model. Matsuyama (2002) proposes a subtle interactive link between the two sub-economies by allowing the two a priori identical players to take two decisions, one in each sub-economy. Assuming players’ actions are pairwise strategic complements and each player’s own actions are substitutes due to a fixed total resource constraint, multiple equilibria arise with the property that the symmetric equilibria are Cournot-unstable while the asymmetric ones are stable. Endogenous heterogeneity in this original approach is predicated on the key postulate that only Cournot-stable equilibria are observable outcomes of this complex game, any of which would involve each agent taking different actions in the ex ante identical sub-economies. In his pioneering work, Matsuyama (2000, 2002, 2004) elaborates on the concept of symmetry breaking in development, international, labor and macro economies, in novel ways that challenge the conventional approach.

The third approach originates in the business strategy and management literature, which has a long-standing tradition of research on firms as complex
organizations and on the sources and dynamics of inter-firm heterogeneity. In
the dominant view, firms operate in such highly complex and ever-changing en-
vvironments that they entertain no hope of ever accumulating enough knowledge
about their world to view it as a strategic game, much less to formulate a pre-
cise strategy for systematic behavior. Rather, firms grope for economic perfor-
ance via a heuristic learning process involving elementary trial and error and
the continual updating of routines and rules of thumb eschewing optimization.
Inter-firm heterogeneity is then simply an inevitable outcome of such inherently
idiosyncratic processes, and firms end up with different heuristic "strategies"
and core capabilities (Nelson and Winter, 1982).

The present paper is an attempt to contribute to this debate along standard
lines of argument in applied game theory and industrial organization. Consider
a two-player symmetric normal-form game characterized by two key properties:
actions form strategic substitutes, and each player’s payoff, though continuous,
admits a robust concavity-destroying kink along the diagonal in action space,
which results in a jump of the reaction correspondence across the 45° line. Such
a game always admits pure-strategy Nash equilibrium points due simply to the
property of global strategic substitutes. Furthermore, due to the jump over the
diagonal, no such equilibrium could ever be symmetric. At any of the possibly
multiple equilibria, which occur in pairs due to the symmetry of the game, oth-
erwise identical agents will necessarily take different equilibrium actions. With
this being the main result of the paper, the same conclusion is also shown to
hold for a class of games with payoffs that are quasi-convex in own action and
submodular on the four-point square formed by the players’ extreme actions
only, reflecting thus global nonconcavity and extreme-point submodularity. As
argued in some detail later on, while relying on a different symmetry-breaking
mechanism, the present approach for two-player games is close in spirit to Mat-
suyama’s (2002) view, and also possesses a nice complementary scope of ap-
lication. Indeed, while the latter study has found, or was even motivated by,
various applications in growth, development and international economics, the
same can be said for the former vis a vis industrial organization.
The present paper may also be motivated in relation to various broad strands of literature in industrial economics dealing in some way with strategic endogenous heterogeneity along lines similar to ours here. The first literature that comes to mind is concerned with product differentiation. In a myriad of two-stage games where each firm chooses a quality level or a location in the first stage, and a price for its product in the second stage, endogenous heterogeneity naturally emerges out of the firms’ perception that identical choices in the first stage would lead to zero profits due to the Bertrand paradox.\(^2\)

The second, extensive literature deals with infinite-horizon industry dynamics allowing for entry and exit. One class of models, exemplified by Jovanovic (1982), postulates perfectly competitive firms for which differences emerge due to *exogenous* idiosyncratic technology shocks. Another class is formed by studies that do generate endogenous heterogeneity in long run dynamics by considering firms that invest in capacity expansion (e.g. Besanko and Doraszelski, 2002) or research and development (e.g. Doraszelski and Satterthwaite, 2008).

Some simpler two-stage models with similar flavor but without entry and exit also generate endogenous differences amongst competing firms. Using a standard set-up in industrial organization, these models feature a long-run investment decision in the first stage and short-run product market competition in the second stage. As first period decisions, Maggi (1996), Reynolds and Wilson (2000) and De Frutos and Fabra (2007) deal with capacity expansion. Amir and Wooders (2000), Mills and Smith (1996), and Amir (2000) consider R&D/product market competition games, with the latter two studies giving rise to equilibrium outcomes with maximal heterogeneity only, i.e. full R&D by one firm and no R&D by the rival. Ireland (1993) considers strategic information provision (or informative advertising). Being those that are most closely related to the unifying approach taken in the present paper, these models are reviewed in some detail below, as direct illustrations of our results.

There are other studies in various areas of applied microeconomics where

---

endogenous heterogeneity emerges in a strategic setting. A partial list follows. Hermelin (1994) develops a two-stage game where firms’ choices of managerial structures take place before product market competition. In a tax competition model, Mintz and Tulkens (1986) exhibit asymmetric tax rates for identical member states for some parameter regions. Another strand of literature, not directly related to our setting, deals with endogenous heterogeneity arising out of hybrid two-stage models of joint ventures wherein firms make a cooperative decision in the first stage (based on joint profit maximization) followed by strategic product market competition in the second stage: See Salant and Shaffer (1999), Long and Soubeyran (2001) and Amir, Evstigneev and Wooders (2003).

While conceived with different motivations, each of these studies might be construed as a context-specific attempt to shed light on the central issue at hand. Taken all together, one might hope that these studies share some general features or a common mechanism for generating symmetry breaking in strategic settings, which could be put forth as another systematic approach. As an alternative motivation, the present paper is an attempt to develop a unifying approach to understanding symmetry-breaking mechanisms in general classes of two-player games, encompassing many of the cited studies. These two-stage models share two key features that are critical for the symmetry-breaking arguments they present. The first is a fundamental nonconcavity in the payoffs, which may be confined to the diagonal in action space or hold globally, and the second is some form of strategic substitutability in first-period actions. A noteworthy aspect of the results is that they rely on critical assumptions that arise naturally, are economically intuitive in their respective contexts, and easy to verify in various applications in industrial organization presented below.

The paper is organized as follows. Section 2 contains the overall set-up. Section 3 provides the results on the exclusive existence of interior asymmetric equilibria for submodular payoff functions. Section 4 presents the results for games with quasi-convex payoffs (in own action) and extreme asymmetric equilibria. Each section contains a summary of the relevant applications the results pertain to. The Appendix provides most of the proofs.
2 The set-up

This section lays out the set-up and general notation for use throughout the paper. Consider a two-player normal form symmetric game $\Gamma$ with common action set $X = [0, c] \subset R$, and payoff functions $F$ and $G : X \times Y \to R$. By symmetry of the game $\Gamma$, the payoff of player 2 is $G(x, y) = F(y, x)$. Except where otherwise indicated, the payoff function of player 1 is always of the form

$$F(x, y) = \begin{cases} 
U(x, y), & x \geq y \\
L(x, y), & x < y
\end{cases}$$  \hspace{1cm} (1)

for some functions $U$ and $L$ with respective domains

$$\Delta_U = \{(x, y) \in [0, c]^2 : x \geq y\} \text{ and } \Delta_L = \{(x, y) \in [0, c]^2 : x \leq y\}.$$

It will be assumed throughout this section that $U$, $L$ and $F$ are jointly continuous functions of the two actions, so in particular that $U(x, x) = L(x, x), \forall x \in [0, c]$. It follows that the best response correspondences (or reaction curves) for players 1 and 2, defined respectively as $r_1(y) = \arg \max \{F(x, y) : x \in [0, c]\}$ and $r_2(x) = \arg \max \{F(y, x) : y \in [0, c]\}$ are well-defined.

As usual, a pure strategy Nash equilibrium, (or PSNE for short), $(x^*, y^*) \in [0, c]^2$ is said to be symmetric if $x^* = y^*$, and asymmetric otherwise. It follows from the symmetry of the game that if $(x^*, y^*)$ is a PSNE, so is $(y^*, x^*)$.

The noncooperative game described may be a simple one-shot game or it may represent the payoffs of a two-stage game as a function of the first period actions, where the unique second stage pure-strategy equilibrium has been substituted in. In the latter case, which actually covers most of the applications of this paper, we obviously restrict consideration to subgame-perfect equilibria and analyze the resulting one-shot game.

The next two sections investigate two separate classes of normal-form symmetric games that always possess asymmetric Nash equilibria and no symmetric Nash equilibria. For each of the two classes, we provide a set of easily verified assumptions establishing both the existence of asymmetric, and the inexistence of symmetric, PSNEs. To illustrate the relevance of our results and the ease of
verification of our sufficient conditions, we provide some detailed illustrations based on previous studies in industrial organization where a special case of our results was derived in a specific economic setting.

The definitions and main results from the theory of supermodular games used in this paper are reviewed in the Appendix in an elementary way, which is sufficient for the purposes of this paper.

3 Endogenous heterogeneity with strategic substitutes and diagonal non-concavity

In this section, we consider a two-player normal-form symmetric game characterized by two key properties. The first is that, conditional on one player using a higher action than the other, the two actions form strategic substitutes. This means that an increase in one player’s strategy lowers the other player’s marginal returns to increasing his own strategy, as long as all the latter’s actions remain larger or smaller than the former’s. The second key property is that each player’s payoff, though jointly continuous in the two actions, admits a fundamental nonconcavity along the $45^\circ$ line, giving rise to a canyon shape along the diagonal. A key consequence of this feature is that a player would never optimally respond to an action of the rival by playing that same action himself.

Taken together, these two properties imply that the actions are globally strategic substitutes, and thus that each best reply is a decreasing correspondence, which in addition has a downward jump over the $45^\circ$ line.$^3$ It follows that a pure-strategy Nash equilibrium (henceforth, PSNE) exists, see Appendix. Hence, no PSNE could ever be symmetric. At any of the possibly multiple equilibria, which occur in pairs due to the symmetry of the game, ex ante identical agents will necessarily take different equilibrium actions.

$^3$We say that a function $f : R \to R$ is increasing (strictly increasing) if $x' > x$ implies $f(x') \geq (>) f(x)$. A correspondence is increasing if all its selections are increasing functions.
3.1 The results

The following assumptions will be needed for the analysis of Section 3, but not of Section 4. The notation is as laid out in Section 2.\(^4\) A full discussion of the assumptions and results is presented at the end of the section. Most of the proofs can be found in the Appendix.

**A1** \(U, L\) are twice continuously differentiable on \(\Delta_U\) and \(\Delta_L\), respectively,\(^5\) and \(F\) is continuous along the diagonal, i.e. \(U(x,x) = L(x,x), \forall x \in [0,c]\).

**A2** \(U, L\) are strictly submodular on the sublattices \(\Delta_U\) and \(\Delta_L\), respectively.

**A3** \(U_1(x^+, x) > L_1(x^-, x), \forall x \in (0,c)\).

**A4** \(U_1(0^+, 0) > 0\) and \(L_1(c^-, c) < 0\).

**A2** says that on either side of the diagonal, but not necessarily globally, each player’s marginal returns to increasing his action decrease with the rival’s action. **A3** holds that each player’s payoff, though globally continuous in the two actions, has a concavity-destroying kink along the diagonal.\(^6\) The role of **A4** is simply to rule out PSNEs at the extreme actions, i.e., \((0,0)\) and \((c,c)\).

These assumptions form a sufficiently general framework to encompass many of the studies mentioned in the Introduction as special cases. Furthermore, all the assumptions are easy to check directly on the primitives of a particular game, as will be seen in the applications below.

The following is the main result of this paper.

**Theorem 3.1** Assume that **A1 – A4** hold. Then the game \(\Gamma\) is of strategic substitutes, has at least one pair of asymmetric PSNEs and no symmetric PSNEs.

---

\(^4\)In addition, throughout the paper, partial derivatives are denoted by a subindex corresponding to the relevant variable, i.e. \(U_1(x,y) = \partial_U(x,y)\) and \(U_2(x,y) = \partial_Y(x,y)\). One-sided derivatives at a point are indicated by a + or − sign as an exponent after the point, e.g. \(U_1(0^+, y)\) is the partial derivative from the right w.r.t \(x\) at \((0,y)\).

\(^5\)On the boundaries of the domains \(\Delta_U\) and \(\Delta_L\), the derivatives are to be understood as one-sided (directional) derivatives.

\(^6\)Smoothness assumptions are not critical, but made purely for convenience. Indeed, Assumption **A3** could be stated using one-sided finite differences.
The idea of the proof is that global submodularity of the payoff function is inherited from the submodularity of its components $U$ and $L$ in the presence of Assumption $A3$, which leads by Topkis’s monotonicity theorem to globally decreasing best replies. $A3$ ensures that the best response correspondences must have a downward jump that crosses over the diagonal.\(^7\)

The prototypical case is depicted in Figure 1.\(^8\) Global submodularity of $F$ gives us the existence of a PSNE via the strategic substitutes property (a result due to Vives, 1990), and the jump across the diagonal (at point $d$) precludes symmetric equilibria. That PSNEs come in pairs is a direct consequence of the symmetry of the game. The complete proof can be found in the Appendix.\(^9\)

Theorem 3.1 does not rule out the existence of multiple pairs of PSNEs. Indeed, the two reaction curves may intersect several times above and below the diagonal. In case of multiple pairs of PSNEs, there will typically be coexistence of pairs of Cournot-stable and pairs of Cournot-unstable PSNEs.\(^10\) Nevertheless, Theorem 3.1 does imply that all of these PSNEs are asymmetric. Thus, the present approach to symmetry-breaking is not at odds with Schelling’s (1960) notion of focalness of PSNEs.

---

\(^7\)This result would be more general if one could use the dual single-crossing property instead of submodularity (Milgrom and Shannon, 1994). There are two reasons that justify our choice of (cardinal) submodularity. The first is that all the economic applications that motivated the present paper use objective functions that are formed by adding separate parts (such as revenue and costs), an operation that need not preserve the single-crossing property. The second reason is that our approach for showing that the overall payoff inherits the submodularity of the separate components $U$ and $L$ does not appear to extend to the single-crossing property.

\(^8\)Note that $x$ is the variable on the vertical axis throughout. This corresponds to analyzing the game from the point of view of player 1 choosing $x$ as a response to $y$. Also, continuity of the reaction curves in each triangle over and below the diagonal is only there for the sake of a clearer figure. It need not hold under Assumptions $A1$- $A4$.

\(^9\)It is important to note that global submodularity of the the overall payoff function is not a primitive assumption on the structure of the game at hand. Indeed, the proof that the overall payoff inherits the assumed submodularity properties of $U$ and $L$ in the presence of assumption $A3$ is surprisingly long. The key difficulty is that Topkis’s convenient cross-partial test for submodularity cannot be invoked, due to the presence of the kink along the diagonal.

\(^10\)For comparative statics properties of such equilibria, see Echenique (2002).
Figure 1: Decreasing reaction curves have a jump along the diagonal, and there is no symmetric equilibrium.

Since payoffs are jointly continuous in actions for this class of games, a symmetric mixed-strategy Nash equilibrium always exists (Dasgupta and Maskin, 1986). Indeed, in contrast to the property of submodularity, even if one were willing to assume that $U$ and $L$ are both strongly concave in own action, the overall payoff function would not inherit the property of global concavity in own action, due to the fundamental non-concavity along the diagonal. Hence, some scope for randomization is always present for players in this class of games.

As a symmetric mixed-strategy Nash equilibrium would be the only focal equilibrium (extrapolating from a well-known term first coined by Schelling, 1960), it may reasonably be advanced as a plausible outcome of such a game. Nevertheless, in actual realizations of the equilibrium randomizations, the players will still end up actually choosing different actions with strictly positive, 11As a consequence, this result is not in conflict with the well-known results on the oddness of the total number of (pure and mixed) Nash equilibria (in a generic sense).
though not full, probability. Hence, given our focus on explaining observed heterogeneity, the approach followed in the present paper need not rule out mixed strategies on a priori grounds. On the other hand, since two-player submodular games are also supermodular games (upon reversing the order on one player’s action set), all mixed-strategy equilibria are unstable for a broad class of natural learning dynamics (Echenique and Edlin, 2004). This instability property makes mixed-strategy equilibria rather unappealing from an applied economist’s perspective, particularly when taken in conjunction with the commonly perceived shortcomings of such equilibria by economists, such as their ex post regret property.

Towards the goal of generating endogenous heterogeneity across economic agents, the present approach is closer in spirit to Matsuyama’s (2002) symmetry-breaking explanation than to the earlier approaches in the extant literature. Both the present approach and Matsuyama’s postulate ex ante identical agents that interact in one economy and identify a combination of features, including a critical complementarity component, that lead to asymmetric PSNEs being the only plausible outcomes of the game, thereby showing how inter-agent heterogeneity is an inherent by-product of strategic interaction in their respective classes of models. Nevertheless, the nature of the mechanisms underlying the two approaches are quite distinct. In Matsuyama’s approach, the prototypical situation would consider a large number of firms deciding in which region or country to invest, or which racial/gender groups to concentrate recruiting efforts on, when there are strategic complementarities in their choices of regions, countries or groups. The instability (in the sense of Cournot dynamics) of steady-state, balanced growth, or equal treatment of different groups generates business cycles, and uneven development across regions or nations, as well as racial and gender inequality. In contrast, the present approach relies on a class of games that posits just two players whose strategic interaction is characterized by two key properties: (i) activity levels form strategic substitutes (conditional on one being larger or smaller than the other), and (ii) the option of matching the activity level of the otherwise identical rival can never be a best-response,
as reflected in a robust diagonal-skipping discontinuity in the reaction curves.

The conjunction of these two key properties leads players to sort themselves out into natural high-activity and low-activity agents, or alternatively into a natural leader and follower pair, in a sense clearly distinct from the sequential-move (Stackelberg) paradigm. (In fact, given the ex ante symmetry of the game, our analysis can provide no equilibrium selection argument as to which player is likely to emerge as a leader or as a follower.) This general mechanism arguably has some potential for explaining in a unified way the critical structures behind the emergence of various aspects of intra-industry heterogeneity, some prominent examples of which are innovation or technological leaders and followers, high-quality and low-quality firms, large and small firms (in the sense of production capacities), high- and low-advertizing firms, etc ... The applications provided next elaborate on some of these specific contexts.

3.2 Some applications in industrial organization

In this section we present examples of economic models that constitute special cases of (some or all features of) the general framework developed above. While the assumptions validating Theorem 3.1 might at first appear somewhat special, they are satisfied in several a priori unrelated studies that have established endogenous heterogeneity in strategic settings in various sub-areas of industrial organization. Going over some of these examples allows us to illustrate the unifying character of our results, the ease of direct verification of our assumptions, and also to provide some contextual interpretations of symmetry-breaking.\footnote{The list of papers reviewed here is not exhaustive. For instance, the taxation model of Mintz and Tulkens (1986) fits our setting (at least for some parameter values), but we omit its description for the sake of brevity. In some studies, asymmetric equilibria are produced via a mechanism similar to ours, in the sense of relying on the same two key properties of strategic substitutes and a critical discontinuity, without being a special case in a formal sense. For instance, the model of organizational design by Hermelin (1994) does not fit our setting in a strict sense, but only because the number of managers is a discrete decision variable.}
3.2.1 R&D investment

In Amir and Wooders (2000), two a priori identical firms with initial unit cost \( c \) are engaged in a two stage game of R&D investment and production. In the first stage, autonomous cost reductions \( x \) and \( y \) for firms 1 and 2, respectively, are chosen. The novel feature of this study is that spillovers are postulated to flow only from the more R&D active firm to the rival, but not vice versa. The effective (post-spillover) cost reductions \( X \) and \( Y \) when \( x \geq y \) are given by:

\[
X = x \text{ and } Y = \begin{cases} 
  x & \text{with probability } \beta \\
  y & \text{with probability } 1 - \beta 
\end{cases}
\]  

Second stage product market competition, be it Cournot or Bertrand, is assumed to have a unique PSNE with equilibrium payoffs given by \( \Pi : [0, c]^2 \to \mathbb{R} \). \( \Pi(x, y) \) is the payoff of the firm whose unit cost is the first argument. Let \( f : [0, c] \to \mathbb{R} \) be a known R&D cost schedule, with \( f'(x) \geq 0 \). Assume the following (in addition to smoothness of \( \Pi \) and \( f \)):

**C1** \( \Pi \) is strictly submodular and \( \Pi_1(x, y) < 0 \) and \( \Pi_2(x, y) > 0 \).

**C2** \( |\Pi_1(x, x)| > |\Pi_2(x, x)|, \forall x \in [0, c] \).

**C3** \( f'(0) < -\beta \Pi_2(c, c^-) - \Pi_1(c^+, c) \) and \( f'(c) > - (1 - \beta) \Pi_1(0^+, 0) \).

The overall payoff of firm 1, \( F(x, y) \), defined as in (1), is given by the difference between its second stage profit and first stage R&D cost.

\[
U(x, y) = \beta \Pi(c - x, c - x) + (1 - \beta) \Pi(c - x, c - y) - f(x)
\]

\[
L(x, y) = \beta \Pi(c - y, c - y) + (1 - \beta) \Pi(c - x, c - y) - f(x)
\]

We can easily check that A1 and A2 indeed hold. \( U(x, x) = L(x, x), \forall x \in [0, c] \), so \( F \) is continuous. A1 can be checked by using the cross-partial test and the fact that \( \Pi(x, y) \) is submodular (C1). Using C1 – C2, and

\[
U_1(x, x) = -[\Pi_1(c - x, c - x) + \beta \Pi_2(c - x, c - x)] - f'(x)
\]

\[
L_1(x, x) = -(1 - \beta) \Pi_1(c - x, c - x) - f'(x)
\]
we see that $U_1(x,x) > L_1(x,x)$, so $A3$ holds. Finally, $A4$ follows from $C3$.

Hence, Theorem 3.1 applies directly to this model.

The key driving force behind asymmetric equilibrium outcomes here is the one-way nature of the spillover process. A firm will always react by performing either less R&D than its rival knowing that it may free ride on the difference in R&D levels, or, in case the rival’s R&D is simply too low, by overtaking it. In this vision, firms will endogenously settle into R&D innovator and imitator roles simply as a reflection of the nature of the R&D spillover process.\footnote{Amir and Wooders (2000) argue that this spillover process is appropriate in industries where the R&D process is well-approximated by a one-dimensional process (i.e. an essentially well laid out sequence of consecutive tests that all firms must follow).}

### 3.2.2 Capacity choice and demand uncertainty

The literature dealing with long-run competition in capacities and short-run product market competition features several studies that relate to the present framework to varying extents. We review these studies briefly here.

De Frutos and Fabra (2007) offer an elegant and general analysis of a common two-period duopoly model where firms choose capacities under demand uncertainty in the first period, and then engage in price competition upon demand realization in the second period. When demand has a continuous distribution, they establish the existence of an asymmetric PSNE, with no symmetric PSNE being possible, under general assumptions. When demand has a discrete distribution, symmetric and asymmetric PSNE may co-exist, but the latter are always present. Thus, under quite general conditions, their model leads to endogenous capacity asymmetries even though firms are ex-ante identical. De Frutos and Fabra provide some interesting intuition behind the twin strategic incentives that lead identical firms to settle into high and low capacity producers. The underlying mechanism under continuous demand relies on the same two key properties and is thus quite closely parallel to the main result of the present paper. Their analysis might alternatively be conducted by directly verifying Assumptions $A1 - A4$ for their model.
Analysing a related variant of the standard two-stage capacity-then-price duopoly model as in De Frutos and Fabra (2007), Reynolds and Wilson (2000) establish that when demand variability exceeds a certain threshold, a symmetric equilibrium cannot exist for a general class of market demand functions (see their Theorem 1(ii) and its Corollary). The underlying reason is the presence of a concavity-destroying kink along the diagonal (in capacity choice space), exactly as reflected in our Assumption A3. Then specializing their model to linear demand with a binary distribution for the unknown demand intercept, Reynolds and Wilson (2000) prove that the only pure-strategy equilibria in capacity choice are asymmetric (see their Theorem 2(ii)). In this case, they show that the reaction curves in capacity choices are downward-sloping and have a jump that skips over the diagonal. It is easy to verify that this specific version of their model satisfies our Assumptions A1 – A4.\(^{14}\)

Deneckere and Davidson (1986) consider a variant of the well-known capacity-then-price duopoly model with deterministic demand of Kreps and Scheinkman (1983), and report a tendency towards asymmetric outcomes. However, despite some similarities, their setting does not directly fit our framework.\(^{15}\)

### 3.2.3 Provision of information

Ireland (1993) presents a two-stage model of information provision in a Bertrand oligopoly. In the first stage, firms decide on the provision of information about the existence of their product, and in the second stage, they compete in prices. Each firm has monopoly power over the consumers who are only informed about

\(^{14}\)Staiger and Wolak (1992) consider a repeated-stage model where the constituent game features capacity and price choices, in a way that makes it a special case of the general model in Reynolds and Wilson (2000). Therefore, the constituent game of Staiger and Wolak also satisfies our Assumption A3, so that symmetric PSNE are ruled out, contrary to what was claimed by the authors (for more on this point, see Reynolds and Wilson, 2000).

\(^{15}\)For one thing, their analysis requires mixed-strategy equilibria in some of the price sub-games, due to the non-existence of PSNEs. On the other hand, the specific model with linear demand at the end of their paper is quite close in spirit to our framework, and satisfies Assumption A3 when the cost of capacity is low.
the existence of its product but not of the rival’s product. As such, whenever
the information coverage choice of one firm is low, the other wishes to opt for
a high information coverage. Hence, profit functions are submodular. The non-
concavity along the diagonal arises from price competition driving profits to zero
whenever the same consumers are informed of the existence of both products,
according to the usual Bertrand mechanism. This paper concludes that no
symmetric PSNE in information provision exists, and that a pair of asymmetric
PSNE may be found. It is easy to show that Assumptions A1 – A4 are verified
in this example and could be used to obtain the asymmetry result directly, or
even to generalize Ireland’s analysis to a broader specification of his model.

4 Games with quasi-convex payoffs

In this section we analyze a class of symmetric games in which payoff functions
are strictly quasi-convex in own action. This leads players to always prefer one
of their extreme actions in response to any strategy of the rival. It follows that
only asymmetric PSNEs involving extreme actions can arise. A dual result is
in Amir et al. (2008), who consider n-player symmetric games on chains with
order-quasi-convex payoffs that are supermodular for extreme actions, and show
existence of symmetric equilibria only.

Given \( y \), define the right lower Dini derivate w.r.to \( x \) at 0 and the left upper
Dini derivate w.r.to \( x \) at \( c \) respectively by (recall that these derivates always
exist in the extended reals for any function, Royden, 1968)

\[
F_1(0^+, y) \triangleq \liminf_{x \downarrow 0} \frac{F(x, y) - F(0, y)}{x} \quad \text{and} \quad F_1(c^-, y) \triangleq \limsup_{x \uparrow c} \frac{F(c, y) - F(x, y)}{c - x}
\]

Theorem 4.1 Consider a symmetric game with action set \([0, c]\) and payoff
function \( F(x, y) \) that is upper semi-continuous and strictly quasi-convex in own
action. Assume that either

\[
F(c, 0) > F(0, 0) \quad \text{and} \quad F(0, c) > F(c, c)
\]
or

\[ F_1(0^+, 0) > 0 \text{ and } F_1(c^-, c) < 0 \]  

(8)

Then, the game has no symmetric PSNE and it has exactly one pair of asymmetric PSNEs given by \{\{(0, c), (c, 0)\}\}.

**Proof.** Since \(F(x, y)\) is u.s.c. in \(x\), \(r_1(y)\) is well-defined. By strict quasi-convexity, against any \(y \in [0, c]\) by Player 2, any \(x \in (0, c)\) yields a strictly lower payoff to Player 1 than either \(x = 0\) or \(x = c\). This implies that \(\forall y \in [0, c]\), \(r_1(y) \in \{0, c\}\). By symmetry, the same conclusion holds for player 2.

Now consider (7). Clearly, \(r_1(0) \in \{0, c\}\) and \(F(c, 0) > F(0, 0) \Rightarrow r_1(0) = c\). Likewise, \(r_1(c) \in \{0, c\}\) and \(F(0, c) > F(c, c) \Rightarrow r_1(c) = 0\). Hence \((c, 0)\) and \((0, c)\) are the only PSNEs of the game.

Finally, consider (8). From strict quasi-convexity, \(F_1(0^+, 0) > 0\) implies that \(F(\cdot, 0)\) is strictly increasing on \([0, c]\), and thus that \(r_1(0) = c\). Likewise, \(F_1(c^-, c) < 0\) implies that \(F(\cdot, 0)\) is strictly decreasing, and thus \(r_1(c) = 0\). Hence \((c, 0)\) and \((0, c)\) are the only PSNEs of the game.

In actual applications, it is often easier to check condition (8) than condition (7), since the former is often a direct consequence of natural Inada-type assumptions on the primitives of a model. Indeed, for models with non-specific functional forms, checking (7) directly is often difficult.

Submodularity in extreme actions is clearly implied by (7). Indeed, adding up the two inequalities in (7) reveals that \(F\) is submodular on the 4-point system \{\{(0, 0), (0, c), (c, 0), (c, c)\}\}, although not on \((0, c)^2\).

Figure 2 illustrates Theorem 4.1.

We now list a number of existing applications of this result. The best-known models that use the same arguments as the present result are those dealing with duopoly with vertically differentiated products (Gabszewicz and Thisse, 1979 and Shaked and Sutton, 1982). Consider two firms engaged in a two-stage game where they make a quality decision in the first stage and a price decision in

\[16\] This result clearly extends to asymmetric games (with the same proof). Since the focus of this paper is on endogenous heterogeneity, we dealt with the symmetric version.
the second stage. Under a few variants of such a basic set-up, the overall payoff of each firm as a function of the two quality decisions, conditional on a Bertrand equilibrium in the price subgame, is strictly convex in own quality. Since the same quality choice would lead to homogeneous goods in the price subgame and thus zero profit, subgame-perfect equilibrium calls for the adoption of maximal quality by one firm and minimal quality by the other firm (the quality space is an exogenous compact interval). It is trivial to verify that the analysis presented in these related papers is a special case of the present result.

Theorem also generalizes the results of Mills and Smith (1996) and Amir (2000), who consider two-stage duopoly games of R&D/product market competition. In the first stage, firms make R&D investment decisions that affect the production costs. In the second stage, firms compete à la Cournot. Under some plausible assumptions, each profit function is strictly convex in own R&D decision, so that extreme asymmetric outcomes form the only equilibria of the game. It is easily shown that the conditions presented in these papers can also
be deduced from the assumptions of Theorem 4.1.

5 Conclusion

This paper has provided simple, easily verified and general conditions on the primitives of a symmetric two-player game that ensures that heterogeneity in a priori identical agents’ behavior necessarily arises at any pure-strategy equilibrium. This paper thus constitutes a contribution to the discussion on the sources of diversity across economic agents and disparities in economic performances. While previous literature stands on arguments related to distinct equilibrium selections and strategic complementarities (Cooper, 1999) or on a mix of strategic substitutability and complementarity and Cournot-stability of equilibria (Matsuyama, 2002), our approach stands on the presence of a fundamental nonconcavity along the diagonal of the payoff function along with some form of strategic substitutability. The earlier strands of literature are more suitable for, and indeed were motivated by, important unequal treatment issues that arise in growth, labor and international economics. The present approach is more appropriate for models arising in various sub-areas of industrial organization, as a way to shed light on the endogenous emergence of leader-follower configurations in various duopoly models. As such, the present approach provides a complementary scope to the extant literature in terms of areas of economic applications, and to some extent also in terms of methodological perspective.

As demonstrated by the many existing studies reported here, for which the present analysis was conceived to provide a unifying framework, the scope for application is both broad and natural. From a methodological perspective, it emerges quite clearly that an approach based on supermodularity is more appropriate for the present framework than the usual concavity approach.

The conclusions reached in the present paper certainly call for further research on the topic. In particular, an extension of the analysis to $n$-player games, along with the concomitant multiplicity in modes of endogenous heterogeneity, would be of substantial interest for industrial economists.
Appendix

6.1 Summary of supermodular/submodular games

We give an overview of the main definitions and results in the theory of supermodular games that are used in this paper, in a simplified setting that is sufficient for our purposes. Details may be found in Topkis (1978, 1998).

Let \( I_1 \) and \( I_2 \) be compact real intervals. \( F: I_1 \times I_2 \to \mathbb{R} \) is (strictly) supermodular\(^{18} \) if \( \forall x_1, x_2 \in I_1, x_2 > x_1 \) and \( \forall y_1, y_2 \in I_2, y_2 > y_1 \) we have \( F(x_2, y_2) - F(x_2, y_1) > F(x_1, y_2) - F(x_1, y_1) \). \( F \) is (strictly) submodular if \(-F\) is (strictly) supermodular.

Let \( F \) be twice continuously differentiable. Then \( F \) is supermodular (submodular) if and only if \( F_{12} = \frac{\partial^2 F}{\partial x \partial y} \geq 0 \) (\( \leq 0 \)) for all \( x, y \). If \( F_{12} = \frac{\partial^2 F}{\partial x \partial y} > 0 \) (\(< 0 \)) for all \( x, y \), then \( F \) is strictly supermodular (submodular).

We now present a dual (and special case) of Topkis’s monotonicity theorem, which is suitable for our use in this paper.

**Theorem 6.1** If \( F \) is continuous in \( x \) and (strictly) submodular in \((x,y)\), then \( \arg\max_{x \in I_1} F(x,y) \) has maximal and minimal (all of its) selections that are decreasing in \( y \in I_2 \).

A two-player game is supermodular (submodular) if both payoff functions are continuous, supermodular (submodular) and both action spaces are compact real intervals. The fixed point theorem associated with this framework is due to Tarski (1955).

**Theorem 6.2** Let \( f: I_1 \times I_2 \to I_1 \times I_2 \) be an increasing function, then \( f \) has a fixed point.

\(^{17}\) Other aspects of the theory that are relevant to the present paper may be found in Topkis (1979), Vives (1990), Milgrom and Roberts (1990) and Amir (1996a), among others.

\(^{18}\) More precisely, this is actually the standard definition of (strict) increasing differences in the literature. For functions on \( \mathbb{R}^2 \), increasing (decreasing) differences is equivalent to supermodularity (submodularity). We shall use only the latter terminology throughout.
This theorem enables one to prove that a supermodular game always has a pure strategy Nash equilibrium. A two-player submodular game becomes a supermodular game upon reversing the order on one player’s action set.

6.2 Proofs of Section 3

The proof of Theorem 3.1 is organized as follows: we begin with proving four preliminary lemmas, and then present the main proof in two steps: first we show existence of PSNE and afterwards that all PSNEs must be asymmetric. Recall that, contrary to standard practice, the variable $x$ is on the vertical axis while $y$ is on the horizontal axis.

The first lemma states that for a small enough square of points in $[0,c]^2$ with two vertices on the diagonal, strict submodularity of $F$ holds.

**Lemma 6.1** Let $(x, x) \in (0, c)^2$ and consider the four points depicted in Figure 3. If $A_1 - A_3$ hold, then for $\alpha > 0$ small enough,

$$L(x,x) - L(x - \alpha, x) < U(x, x - \alpha) - U(x - \alpha, x - \alpha)$$

or equivalently,

$$F(x,x) - F(x - \alpha, x) < F(x, x - \alpha) - F(x - \alpha, x - \alpha).$$

**Proof.** Take any diagonal point $(x, x) \in (0, c)^2$. It follows from $A_3$ and $A_1$ that given $\varepsilon = U_1(x^+, x) - L_1(x^-, x)$, there is $\hat{\alpha} > 0$ such that for all $\alpha < \hat{\alpha}$,

$$L(x - \alpha, x - \alpha) - L(x - 2\alpha, x - \alpha) + \varepsilon/2 \leq U(x, x - \alpha) - U(x - \alpha, x - \alpha).$$

From $A_2$ we know that

$$L(x - \alpha, x) - L(x - 2\alpha, x) < L(x - \alpha, x - \alpha) - L(x - 2\alpha, x - \alpha).$$

Since $L_1(x, y)$ is continuous in $x$ for fixed $y$ (from $A_1$), it follows that given the above $\varepsilon$, there is $\pi > 0$ such that for all $\alpha < \pi$,

$$|(L(x - \alpha, x) - L(x - 2\alpha, x)) - (L(x, x) - L(x - \alpha, x))| < \varepsilon/2.$$
Recapitulating, for all $\alpha < \pi \wedge \tilde{\alpha}$, we have

$$L(x, x) - L(x - \alpha, x) < L(x - \alpha, x) - L(x - 2\alpha, x) + \varepsilon/2 \text{ by (12)}$$

$$< L(x - \alpha, x - \alpha) - L(x - 2\alpha, x - \alpha) + \varepsilon/2 \text{ by (11)}$$

$$\leq U(x, x - \alpha) - U(x - \alpha, x - \alpha) \text{ by (10)}$$

The inequality with the two outer terms yields (9). □

The next lemma extends the property of submodularity of $F$ from small squares to any square with two vertices on the diagonal.

**Lemma 6.2** For any square with two vertices, $(z, z)$ and $(x, x)$, on the diagonal, such as depicted in Figure 4, we have, with $x > z$,

$$F(x, x) - F(z, x) < F(x, z) - F(z, z).$$

**Proof.** Consider the square depicted in figure 4 and divide it into rectangles, each of which has height equal to the original height of the square and length not larger than $\alpha$, as defined in Lemma 6.1. We will now show that $F$ is supermodular for the four-point system formed by the vertices of each such rectangle, and then we extend the conclusion to the whole square.
Let \( x - z = k\alpha \), where \( k \in \mathbb{R} \), and \( \alpha > 0 \) is small enough. Now consider the rectangle defined by the vertices \((x, x), (x, x - \alpha), (z, x)\) and \((z, x - \alpha)\). From Lemma 6.1 we know that

\[
F(x, x - \alpha) - F(x - \alpha, x - \alpha) > F(x, x) - F(x - \alpha, x)
\]

Also, from A2 we know that

\[
F(x - \alpha, x - \alpha) - F(z, x - \alpha) > F(x - \alpha, x) - F(z, x)
\]

since all the points belong to \( \Delta_L \). Adding these two inequalities we obtain that

\[
F(x, x - \alpha) - F(z, x - \alpha) > F(x, x) - F(z, x)
\]

(13)

Repeating the procedure, consider the rectangle defined by: \((x, x - \alpha), (x, x - 2\alpha), (z, x - \alpha), (z, x - 2\alpha)\). From A2 we know that:

\[
F(x, x - 2\alpha) - F(x - \alpha, x - 2\alpha) F(x, x - \alpha) - F(x - \alpha, x - \alpha)
\]

since all the points belong to \( \Delta_U \). Likewise,

\[
F(x - 2\alpha, x - 2\alpha) - F(z, x - 2\alpha) F(x - 2\alpha, x - \alpha) - F(z, x - \alpha)
\]

since all the points belong to \( \Delta_L \). Using Lemma 6.1 we know that

\[
F(x - \alpha, x - 2\alpha) - F(x - 2\alpha, x - 2\alpha) > F(x - \alpha, x - \alpha) - F(x - 2\alpha, x - \alpha)
\]

Adding the three inequalities we obtain:

\[
F(x, x - 2\alpha) - F(z, x - 2\alpha) > F(x, x - \alpha) - F(z, x - \alpha)
\]

\[
> F(x, x) - F(z, x) \text{ by (13)}.
\]

We can repeat this argument \( k \) times until we get a rectangle whose width is not bigger than \( \alpha \). Once again we use A2 and Lemma 6.1 to show that submodularity holds for this rectangle as well and

\[
F(x, z) - F(z, z) \geq F(x, x) - F(z, x).
\]

Hence submodularity holds for any square with 2 vertices on the diagonal. ■
The following Lemma establishes that the analysis of submodularity on any four-point rectangle in $[0,c]^2$ can be reduced to the analysis of submodularity on squares with 2 vertices on the diagonal.

**Lemma 6.3** If $A_1$ and $A_2$ hold, then $F$ is submodular on $[0,c]^2$.

**Proof.** Due to the kink along the diagonal, one cannot invoke Topkis’s simple cross-partial test to verify submodularity of $F$. Instead, we use the definition of submodularity for any four-point rectangle in $[0,c]^2$. If the rectangle is completely contained in either $\Delta_U$ or $\Delta_L$, then submodularity follows from $A_2$. Every other situation can be reduced by adding sub-rectangles, each of which lying fully in either $\Delta_U$ or $\Delta_L$, to the situation depicted in Figure 3 as we now show. Consider the case of Figure 5 with the four points $(x,z)$, $(z,z)$, $(x,y)$, $(z,y)$ as shown. With $z < x < y$, we know from $A_2$ that, since $F(x,y) = U(x,y)$ on $\Delta_U$, we have

$$F(x,x) - F(z,x) > F(x,y) - F(z,y).$$

From Lemma 6.2, submodularity holds for the vertices of the square $(x,x)$, $(x,z)$, $(z,z)$, $(z,x)$, hence we have

$$F(x,z) - F(z,z) > F(x,x) - F(z,x).$$
Figure 5: If $F$ satisfies submodularity on the square on the diagonal, this implies it satisfies submodularity on the rectangle.

Adding the two inequalities yields

$$F(x, z) - F(z, z) > F(x, y) - F(z, y),$$

which is just the definition of submodularity for $(x, z)$, $(z, z)$, $(x, y)$ and $(z, y)$.

It can be shown via analogous steps that the submodularity of $F$ for any other configuration of four points can be reduced to showing submodularity on squares with two vertices on the diagonal. The details are left out. 

The next result allows us to conclude that the two reaction curves always admit a unique discontinuity that skips over the diagonal, a key step for our endogenous heterogeneity result.

**Lemma 6.4**  Given $A1 - A3$, there exists exactly one point $d \in (0, c)$, such that $r_i(d^-) > d > r_i(d^+), i = 1, 2$.

**Proof.** From Topkis’s monotonicity theorem and Lemma 6.3, all the selections from the best reply correspondences are decreasing. From the general properties of monotone functions, both the right limit $r_i(x^+)$ and the left limit $r_i(x^-)$ exist at any point $x$ for any selection of $r_i$, and are independent of the selection.
Specifically, for any selection \( r_i, r_i(x^+) = \mathcal{R}(x) \) and \( r_i(x^-) = \mathcal{L}(x) \), where \( \mathcal{R} \) and \( \mathcal{L} \) denote the maximal and minimal selections of \( r_1 \).

From assumption \( A_3 \), we know that \((0, 0) \notin \text{Graph}_{r_i} \) and \((c, c) \notin \text{Graph}_{r_i} \) (i.e. \( r_i \) does not go through \((0, 0) \) or \((c, c) \)). These two properties imply that \( r_i \) cannot be identically 0 or \( c \).

We next show that the reaction correspondence \( r_1 \) cannot ever intersect the 45° line at an interior point, i.e. in \((0, c)\). The generalized first order condition for a maximum of \( F \) (say) to occur at a point \((x, x)\) with \( x \in (0, c) \), which applies even in the absence of differentiability, is that \( U_1(x^+, x) \leq L_1(x^-, x) \).

Assumption \( A_3 \) rules out this possibility. Hence no \( x \in (0, c) \) can ever be a best reply to itself, meaning that the reaction correspondences do not cross the 45° line at any interior point.

Since \( r_1 \) starts strictly above 0 (for \( y = 0 \)) and ends strictly below \( c \) (for \( y = c \)), the above properties of \( r_1 \) imply that there exists exactly one \( d \in (0, c) \) such that \( r_1(d^-) = \mathcal{R}(d) > d > r_1(d^+) = \mathcal{L}(d) \). In words, \( r_1 \) must have a downward jump that skips over the diagonal as in Figure 1.

Using Lemmas 6.3 and 6.4 we can now prove Theorem 3.1.

**Proof of Theorem 3.1.**

From Lemma 6.3 we have the global submodularity of the payoff function, which guarantees that a PSNE exists (Vives, 1990).

Consider now the behavior of the reaction curves in the area \( \Delta_U \). The same conclusion follows for \( \Delta_L \) by symmetry. It is easy to verify that (any selections of) the following restricted reaction curves: \( r_1|_{\Delta_U}(y) : [0, d] \rightarrow [d, c] \) and \( r_2|_{\Delta_U}(x) : [d, c] \rightarrow [0, d] \) take values in the shown ranges when restricted to the given domains. Also, both restricted mappings are decreasing as implied by Lemma 6.3. Define the mapping \( B : [d, c] \rightarrow [d, c], B(x) = r_1|_{\Delta_U} \circ r_2|_{\Delta_U}(x) \), which is increasing given that each of \( r_1|_{\Delta_U} \) and \( r_2|_{\Delta_U} \) is decreasing. From Tarski’s fixed point theorem, we know that there exists \( \bar{x} \) such that \( B(\bar{x}) = r_1|_{\Delta_U} \circ r_2|_{\Delta_U}(\bar{x}) \), therefore \((\bar{x}, r_2(\bar{x})|_{\Delta_U})\) is a PSNE. Hence, there must exist at least one pair of asymmetric PSNEs. From Lemma 6.4, there is no symmetric
PSNE in $[0, c]^2$.

References


